ORBITS IN DYNAMICAL SYSTEMS DEFINED BY GROUPS

By

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ABSTRACT

The aim of this paper is to study some basic properties of the orbits of elements in dynamical systems defined by groups. The relation between orbit of an element and the orbit of its inverse element has been established. The nature of the orbit of the identity element in the dynamical system has been studied and a singleton set containing the identity element itself has been obtained. It is observed that the set of all fixed points in the dynamical system is strongly invariant (S-invariant) and is a closed subgroup of the group. By using the topological conjugacy between dynamical systems defined by groups, a relation between the orbit of an element and the orbit of its image element in the other dynamical system has been obtained. Finally, we obtained that if an element is in the kernel of the homomorphism, then it must be eventually fixed at the identity element of the group.

Keywords: Discrete Dynamical System, Homomorphism, Periodic Point, Fixed Point, Trajectory, Orbit.

INTRODUCTION

Topological dynamics is a very active field in pure and applied mathematics that involves tools and techniques from many areas such as analysis, geometry and number theory and has applications in many fields as physics, astronomy. A discrete dynamical system can be defined by taking a non-empty set X with some structure and a function $f:X\to X$. If the non-empty set X is a well-known algebraic structure, then it is interesting to study the nature of the orbits of the elements of the dynamical system and to get the relationships between these orbits. The intention is to study the orbits of the elements of dynamical systems defined by groups and to establish the relationship between orbit of an element and the orbit of its inverse element in the system. It is obtained that the orbit of the identity element is a singleton set containing the identity element only.

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1. Preliminaries

1.1 Discrete Dynamical System (Block & Coppel, 1992)

Let G be a nonempty set having some structure (Algebraic or Topological or Measure theoretic) and $f:G\to G$ be a structure preserving function. Then the ordered pair (G,f) is a discrete dynamical system.

Throught this paper, by a dynamical system we mean an ordered pair (G,f) where G is a group and $f:G\to G$ is a continuous homomorphism (Devaney, 1989).

1.2 Trajectory

In the dynamical system (G,f) the trajectory of an element $a\in G$ is the sequence a, $f(a)$, $f^2(a)$, $f^3(a)$,-----.

The range of the above sequence is called the orbit of the element $a\in G$.

1.3 Fixed Point

In the dynamical system (G,f) an element $a\in G$ is called a fixed point if $f(a)=a$.

1.4 Periodic Point (Block & Ledis, 2014)

In the dynamical system (G,f) an element $a\in G$ is a periodic point if $f^n(a)=a$ for some positive integer n. The least value of 'n' is called the period of 'a'.

1.5 Eventually Fixed Point

In the dynamical system (G,f) an element $a \in G$ is a eventually fixed point of f if $f^n(a)=p$ for some positive integer n where p is a fixed point.

1.6 Invariant Set

A subset A of a dynamical system (G,f) is invariant if $f(A) \subset A$.

1.7 Strongly Invariant Set (S-Invariant Set)

A subset A of a dynamical system (G,f) is S-invariant if f(A)=A.

1.8 Topological Conjugacy

Let $f:X \to X$, $g:Y \to Y$ be two functions. We say that f,g are topologically conjugate if there exists a homeomorphism $h:X \to Y$ such that hof=gof.

The following basic results are useful in proving the theorems obtained in the present paper. We state them without proof.

Result 1 (Ramachandram, 2012): If $f:G \to G$ is a homomorphism, then for all $a \in G$ $f(a^{-1}) = [f(a)]^{-1}$.

Result 2 (Ramachandram, 2012): If $f:G \to G$ is a homomorphism, then for all $a \in G$ $f^n(a^{-1}) = [f^n(a)]^{-1}$.

2. Main Results

Theorem 1: (Brin & Stuck, 2002)

In the dynamical system (G,f) the orbit of the identity element $e \in G$ is the singleton set that is Orb(e,f)={e}.

Proof: Trivial.

Theorem 2: (Walter, 1953)

In the dynamical system (G,f), $a \in Orb(b,f) \Leftrightarrow af^n(b)$, for some integer n.

Proof: Trivial.

Theorem 3:

In the dynamical system (G,f), if A is a subset of G such that $f(G) \subset A \subset G$, then A is invariant with respect to f.

Proof: Suppose $A \subset G \Rightarrow f(A) \subset f(G) \Rightarrow f(A) \subset A$. Hence, A is an invariant set in G.

Theorem 4:

In the dynamical system (G,f), the set of all fixed points in G is S-invariant.

Proof: Let A be the set of all fixed points in G. So, $A=\{a \in G / f(a)=a\}$. Let $f(a) \in f(A)$.

Then, $a \in A \Rightarrow f(a)=a \Rightarrow f(a) \in A$. If $a \in A$ then $f(a) \in f(A) \Rightarrow a \in f(A)$.

Hence, f(A)=A.

Theorem 5:

In the dynamical system (G,f) the set of all fixed points in G is a closed subgroup of G.

Proof: Let H be the set of all fixed points in G and 'e' be the identity element in G.

Since 'e' is a fixed point, $H \neq \emptyset$. Suppose 'a' is a limit point of H. There is a sequence $\{a_n\}$ of elements of H such that $\lim_{n\to\infty} a_n = a$.

Since 'f' is a continuous function, we get,

$$f(\lim_{n\to\infty} a_n) = f(a) \Rightarrow \lim_{n\to\infty} f(a_n) = f(a) \Rightarrow \lim_{n\to\infty} a_n = f(a)$$

By the uniqueness of limit of a sequence we get that f(a)=a. This shows that 'a' is a fixed point in G and hence H is a closed subset of G.

Let $a,b \in H \Rightarrow f(a)=a, f(b)=b$.

Since 'f' is a homomorphism, $f(aob)=f(a)of(b)=aob$ and $f(a)^{-1}=(f(a))^{-1}=a^{-1}$.

Hence H is a closed subgroup of G.

The following is a well known theorem from Block and Ledis (2014). We state the theorem without proof.

Theorem 6: (Devaney, 1989)

Let F and G be topological groups, $f:F\to F$, $g:G\to G$ and $\tau:F\to G$ be a topological conjugacy of f and g. Then,

(a) $\tau^{-1}:G\to F$ is a topological conjugacy.

(b) $\tau of^{n}=g^{n}o\tau$ F for all natural numbers n.

(c) p is a periodic point of f if and only if $\tau(p)$ is a periodic point of g. Further, the prime periods of p and $\tau(p)$ are identical.

(d) If 'p' is a periodic point of f with stable set $W^{s}(p)$, then the stable set of $\tau(p)$ is $\tau(W^{s}(p))$.

(e) The periodic points of 'f' are dense in F, if the periodic points of 'g' are dense in G.

(f) 'f' is topologically transitive on F if and only if g is topologically transitive on G.

(g) 'f' is chaotic on F if and only if g is chaotic on G.

In view of above theorem we can prove the following theorem.

Theorem 7:

Let F and G be topological groups, $f:F\to F$, $g:G\to G$ be two continuous homomorphisms and $\tau:F\to G$ be a topological conjugacy of f and g. Then,

(1) $a\in Orb(x,f)\Leftrightarrow\tau(a)\in Orb(\tau(x),g)$

(2) $a\in\omega(x,f)\Leftrightarrow\tau(a)\in\omega(\tau(x),g)$

(3) $x\in Kerf\Rightarrow\tau(x)\in Kerg$

(4) $Ker\tau\subset Kerf$. Moreover the equality holds if g is a one-one function.

Proof:

(1) Let $a\in Orb(x,f)\Leftrightarrow a=f^{n}(x)$ for some $n\geq 0$

$\Leftrightarrow\tau(a)=(\tau f^{n}(x))$

$\Leftrightarrow\tau(a)=g^{n}(\tau(x))$

$\Leftrightarrow\tau(a)\in Orb(\tau(x,g))$

(2) Let $a\in\omega(x,f)\Rightarrow a$ is a limit point of Orb(x,f)

$\Rightarrow$There exists a subsequence $\{f^{n_k}(x)\}$

such that, $\lim_{n_k\to\infty} f^{n_k}(x)=a$

$$\Rightarrow\tau\left(\lim_{n_k\to\infty} f^{n_k}(x)\right)=\tau(a) \quad \text{(By the continuity of } \tau\text{)}$$

$$\Rightarrow\lim_{n_k\to\infty}\tau\left(f^{n_k}(x)\right)=\tau(a)$$

$$\Rightarrow\lim_{n_k\to\infty} g^{n_k}\left(\tau(x)\right)=\tau(a)$$

$$\Rightarrow \tau(a) \in \omega(\tau(x), g)$$

Conversely, suppose $\tau(a) \in \omega(\tau(x), g)$

$\Rightarrow$There exists a subsequence $\{g^{n_k}(\tau(x))\}$ such that, $\lim_{n_k \to \infty} g^{n_k}(\tau(x)) = \tau(a)$

$$\Rightarrow \lim_{n_k \to \infty} \tau(f^{n_k}(x)) = \tau(a)$$

$$\Rightarrow \lim_{n_k \to \infty} f^{n_k}(x) = a$$

$$\Rightarrow a \in \omega(x, f)$$

(3) Let $x \in \text{Ker} f \Rightarrow f(x) = e$

$\Rightarrow \tau(f(x)) = e'$

$\Rightarrow g(\tau(x)) = e'$

$\Rightarrow \tau(x) \in \text{Ker} g$

(4) Let $x \in \text{Ker}\tau \Rightarrow \tau(x) = e'$

$\Rightarrow g(\tau(x)) = e'$

$\Rightarrow \tau(f(x)) = e' = \tau(e)$

$\Rightarrow f(x) = e$

$\Rightarrow x \in \text{Ker} f$

This proves that $\text{Ker}\tau \subseteq \text{Ker} f$

If the function 'g' is one-one then, $x \in \text{Ker} f \Rightarrow f(x) = e$

$\Rightarrow \tau(f(x)) = e'$

$\Rightarrow g(\tau(x)) = e' = g(e')$

$\Rightarrow \tau(x) = e'$

$\Rightarrow x \in \text{Ker}\tau$

This shows that the equality holds if the function 'g' is a one-one function.

Theorem 8:

If $\tau: F \to G$ is a topological conjugacy, then $x \in \text{Ker}\tau \Rightarrow x$ is eventually fixed at the identity element of G.

Proof: Let $x \in \text{Ker}\tau \Rightarrow \tau(x) = e'$. Hence the element 'x' is eventually fixed at e', the identity element of G.

Conclusion

In the present work, some basic properties of the orbits of elements in a dynamical system defined by groups has been studied. The relationship between the orbit of an element and the orbit of the inverse element has been established. The set of all fixed points in G obtained is S-invariant and is a closed subgroup of the given group.

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