ON PRIME L-FUZZY SUBALGEBRAS

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ABSTRACT. In this paper, we introduce the concepts of L-fuzzy subalgebras and a prime L-fuzzy subalgebra and extend the results for prime L-fuzzy subalgebras of a general algebra when the truth values of taken from a general frame.

Keywords : Frame, algebra of type $\mathscr{F}$, subalgebra, L-fuzzy subalgebra, prime subalgebra.

1. INTRODUCTION

The concept of prime fuzzy ideal was introduced by U.M.Swamy and K.L.N. Swamy in rings and by U.M. Swamy and D.V. Raju in Lattices and later B.V.N. Koguep, C.N. Kuimi and C.Lele discussed certain properties of prime fuzzy ideals of Lattices, when the truth values are taken from the interval $[0, 1]$ of real numbers.

Throughout this paper A always stands for an algebra of some fixed type $\mathscr{F}$ and L denotes a frame, that is L is a complete lattice satisfying the infinite meet distributivity.

2. PRELIMINARIES

We briefly recall certain elementary concepts and notations from the theory of partially ordered sets and lattices. A binary relation $\leqslant$ on a set X is called a partial order on X if it is reflexive, anti-symmetric and transitive. A pair $(X, \leqslant)$ is called a partially ordered set or simply poset if X is a nonempty set and $\leqslant$ is a partial order relation on X. A poset $(X, \leqslant)$ is called a lattice (complete lattice) if every nonempty finite subset (respectively, every arbitrary subset) of X has greatest lower bound and least upper boun in X which are respectively called infimum and supremum also; for any subset A of X, we write inf A or glb A or $\wedge A$ or $\wedge_{a\in A} a$ for

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the greatest lower bound (or infimum) of A and sup A or lub A or $\vee A$ or $\vee_{a\in A}$ for the least upper bound (or supremum) of A. If $A = \{a_1, a_2, ...a_n\}$, then we write for the inf A and $\vee_{i=1}^{\infty} a_i$ or $a_1 \vee a_2 \vee ... \vee a_n$ for the sup A.

If $(L, \leqslant)$is a lattice, then $a \wedge b = inf\{a, b\}$ and $a \vee b = sup\{a, b\}$ give two binary operations $\wedge$ and $\vee$ on L which are both associative, commutative and idempotent and satisfythe absorption laws $a \wedge (a \vee b) = a$ conversely if $\wedge$ and $\vee$ are binary operations on a nonempty set L satisfying all the above properties and if the partial order $\leqslant$ on L defined by $a \leqslant b \iff a = a \wedge b(\iff a \vee b = b)$, then $(L, \leqslant)$ is a lattice elementa in a poset $(L, \leqslant)$ is called the smallest (greatest) element if $a \leqslant x$ (respectively $x \leqslant a$) for all $x \in L$. The smallest and greatest elements, if they exist, are usually dentoed by 0 and 1 respectively. A poset is called bounded if it has both smallest and greatest elements. A complete lattice is necessarily bounded. Logically, the infimum and supremum of the empty subset of a poset, if they exist, are respectively the greatest element and smallest element. A complete lattice $(L, \leqslant)$ is called a frame, if it satisfies the infinite meet distributive law; that is, $a \wedge (supX) = sup\{a \wedge x \mid x \in X\}$ for all $a \in L$ and $X \leqslant L$. It is known that a complete lattice $(L, \leqslant)$ is a frame if and only if, for any a and $b \in L$, there exists a largest element, denoted by $a \to b$, in L such that $x \wedge a \leqslant b \iff x \leqslant a \to b$ for all $x \in L[]$. A poset $(P, \leqslant)$ is called a totally ordered set , if for any a and $b \in P$ either $a \leqslant b$ or $b \leqslant a$. A subset C of a poset $(P, \leqslant)$ is called a chain in P if $(C, \leqslant)$ is totally ordered.

Definition 2.1. An L-fuzzy subalgebra B of A is called proper if A is not the constant map T, that is if $B(a) \neq 1$ for some $a \in A$.

Definition 2.2. A proper L-fuzzy subalgebra B of A is called a prime L-fuzzy subalgebra if for any L-fuzzy subalgebra F and G of A,
$F \wedge G \leqslant B \implies F \leqslant P$ or $G \leqslant P$.

For any subalgebras B of A and $\alpha \in L$, we have an L-fuzzy subalgebra α_B of A defined by

$$\alpha_B(a) = \begin{cases} 1 & \text{if } a \in B \\ \alpha & \text{if } a \notin B \end{cases}$$

and that α_B is called the $\alpha-$level fuzzy subalgebra corresponding to B.

Theorem 2.3. Let A be an algebra of type F and L a frame. Let B be a subalgebra of A and $\in L$. Then the $\alpha-$level fuzzy subalgebra α_B is a prime L-fuzzy subalgebra of A if and only if B is a prime subalgebra of A and is a prime element of L.

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Proof. First observe that α_B is a proper $L-$fuzzy subalgebra of A if and only if $B \neq A$ and $\alpha \neq 1$. Therefore, we can assume that $B \neq A$ and $\alpha \neq 1$. Then α_B is proper. Now, suppose that B is a prime subalgebra of A and α is a prime element in L. Let F and G be $L-$fuzzy subalgebras of A such that $F \nleq \alpha_B$ and $G \nleq \alpha_B$. Then there exist elements a and b in A such that $F(a) \nleq \alpha_B$ and $G(b) \nleq \alpha_B(b)$. Then $\alpha \notin B, b \notin B, F(a) \nleq \alpha$ and $G(b) \nleq \alpha$. Since α is a prime element in L, we get that

$$F(a) \wedge G(b) \nleq \alpha \tag{2.1}$$

Let $< a >$ and $< b >$ be the subalgebras of A generated by a and b respectively. Also since B is a prime subalgebra of A and $a \notin B$ and $b \notin B$, we get that $< a > \cap < b > \nsubseteq B$ and hence we can choose $x \in < a > \cap < b >$ such that $x \notin B$. Since a is an element in the $F(a)-$cut $F_{F(a)}$ and $F_{F(a)}$ is a subalgebra of A, we get that

$$x \in < a > \subseteq F_{F(a)} \text{ and hence } F(a) \leqslant F(x).$$

Similarly, $x \in < b > \subseteq G_{G(b)}$ and hence $G(b) \leqslant G(x)$. Therefore $F(a) \wedge G(b) \leqslant F(x) \wedge G(x) = (F \wedge G)(x)$. From (1.1), it follows that

$$(F \wedge G)(x) \nleq \alpha = \alpha_B(x) \text{ (since } x \notin B)$$

and therefore $F \wedge G \leqslant \alpha_B$.

Thus α_B is a prime $L-$fuzzy subalgebra of A.

Conversely suppose that $\alpha_B$is a prime $L-$fuzzy subalgebra of A. We already have that $B \neq A$ and $\alpha \neq 1$. Let C and D be subalgebras of A such that $C \cap D \subseteq B$. Then by Theorem 1.....,

$$\alpha_C \cap \alpha_D = \alpha_{C \cap D} \leqslant \alpha_B$$

and hence $\alpha_C \leqslant \alpha_B$ or $\alpha_D \leqslant \alpha_B$ which implies that $C \subseteq B$ or $D \subseteq B$. Therefore B is a prime subalgebra of A. Next, let β and $\gamma \in L$ such that $\beta \wedge \gamma \leqslant \alpha$. Then

$$\beta_B \wedge \gamma_B = (\beta \wedge \gamma)_B \leqslant \alpha_B$$

and hence $\beta_B \leqslant \alpha_B$ or $\gamma_B \leqslant \alpha_B$. From this, it follows that $\beta \leqslant \alpha$ or $\gamma \leqslant \alpha$. Therefore α is a prime element in L. □

Theorem 2.4. Let A be an algebra of type $\mathscr{F}$ which contains at least one nullary operation symbol. Let P be a proper $L-$fuzzy subalgebra of A. Then P is prime if and only if the following are satisfied.

(1) P assumes exactly two values

(2) For any $a \in A$, either $P(a) = 1$ or $P(a)$ is a prime element in L

(3) $\{a \in A \mid P(a) = 1\}$ is a prime subalgebra of A.

Theorem 2.5. Let A be an algebra of type $\mathscr{F}$ which contains at least one nullary operation symbol. Then an $L-$fuzzy subalgebra P of A is prime if and only if $P = \alpha_B$, for some prime subalgebra B of A and prime element α in L.

Theorem 2.6. Let A be an algebra of type $\mathscr{F}$ and L a frame. For any α and $\beta \in L$ and proper subalgebra B of C of A,

$$\alpha_B \leqslant \beta_C \iff B \subseteq C \text{ and } \alpha \leqslant \beta,$$

where α_B and β_C are $\alpha-$level and $\beta-$level fuzzy subalgebras of A corresponding to B and C.

Proof. Suppose that $\alpha_B \leqslant \beta_C$. Then, for any $x \in B$,

$$1 = \alpha_B(x) \leqslant \beta_C(x) \text{ and hence } \beta_C(x) = 1$$

so that $x \in C$. Therefore $B \subset C$. For any $y \in A - C$(there is one such, since C is proper), we have $y \notin B$ and $\alpha = \alpha_B(y) \leqslant \beta_C(y) = \beta$.
Thus $\alpha \leqslant \beta$ and $B \subseteq C$. Conversely, suppose that $B \subseteq C$ and $\alpha \leqslant \beta$. For any $a \in A$,

$$a \in B \implies a \in C \text{ and } \alpha_B(a) = 1 = \beta_C(a)$$
$$a \notin B \implies \alpha_B(a) = \alpha \leqslant \beta \text{ or } 1 = \beta_C(a)$$

and hence $\alpha_B \leqslant \beta_C$.

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