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On The Algebraic Lattices

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Abstract

We known that the interval [0, 1] of real numbers is insufficient to have the truth values of general fuzzy statements. In this paper we discuss an important class of lattices which are most suitable to contain the truth values of almost all the fuzzy statements.

Key words: Fuzzy statements, truth values, Lattices, complete lattices, distributivity,
Infinite meet distributivity.

Introduction

It was Goguen [1] who first realized that the closed unit interval [0, 1] of real numbers is not sufficient to have the truth values of general fuzzy statements. The conventional (or ordinary or crisp) statements have truth values in the two - element set {0, 1}, where 0 and 1 stand for 'False' and 'True' respectively, while certain fuzzy statements have truth values in the unit interval [0, 1], which is as infinite bounded totally ordered set under the usual ordering of real numbers. However Goguen [1] pointed out that [0, 1] is insufficient to have the truth values of general fuzzy statements.

For example, let us consider the statement 'India is a good country'. This is a fuzzy statement, since 'being good' is fuzzy. The truth value of this statement is not a real number in the interval [0, 1]. Being good may have several components; good in educational standards, good in literacy literacy among the citizens, good in political awareness, good in medical felicities, good in public transport system etc. The truth value corresponding to each component may be a real number in [0, 1]. If n is the number of such components under consideration, then the truth value of the statement 'India is a good country' is an n – tuple of real numbers in [0, 1], which is an element in $[0,1]^n$. If C is the collection of all countries on this earth and G is the collection of good countries, then G is not a subset of C, but it is a fuzzy subset of C, since 'being good' is fuzzy. That is, G can be considered as a

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function of C into a set like $[0,1]^n$, for some positive integer n.

It is well known that the interval [0, 1] of real numbers is a totally ordered set under the usual ordering of real numbers; while $[0,1]^n$, when $n > 1$, is not totally ordered under the coordinatewise ordering. However, $[0,1]^n$ satisfies certain rich lattice theoretic properties; namely, it is a complete lattice satisfying the infinite meet distributivity. For this reason, U. M. Swamy and D.V. Raju [5 and 6] initiated that complete lattices satisfying the infinite meet distributivity are the most suitable candidates to have the truth values of general fuzzy statements. In this paper, we make a thorough discussion on the above type of lattices.

1. DISTRIBUTIVITY IN LATTICES

A partially ordered set (poset) is a pair $(P,\leq)$, where P is a nonempty set and $\leq$ is a partial order (that is, reflexive, transitive and antisymmetric binary relation) on P. A poset $(L,\leq)$ is said to be a lattice if every two - element subset (and hence any nonempty finite subset) of L has greatest lower bound $(glb)$and least upper bound (lub)in L. If $(L,\leq)$is a lattice and $a \text{ and } b \epsilon L$, then the glb and lub of $\{a,b\}$are denoted by $ab \text{ and } ab$ respectively and and become commutative, associative and idempotent binary operations on L satisfying the absorption laws $a(ab) = a = a(ab))$. Equivalently, a lattice can be viewed as a triple $(L,,)$ where L is a nonempty set and and are binary operations on L which are commutative, associative and idempotent and satisfy the absorption laws; and, in this case, the partial order on L is defined by

$$a \leq b \Leftrightarrow a = ab \Leftrightarrow ab = b.$$

In general, for any subset $\times$ of a poset $(L,\leq)$, if the greatest lower bound (least upper bound) of $\times$ exists in L, then it is denoted by $glb \times$ or $\inf \times$ or $^{x}_{x\epsilon X}$ (respectively, $lub \times$ or sup $\times$ or $^{x}_{x\epsilon X}$). The binary operations ^ and v are called meet and join respectively.

Definition 1.1 Let $(L,,)$ be a lattice.

1. L is said to be distributive over

   if $a(b\ \ c) = (a\ b)\ (a\ \ c)$

   for all $a, b \text{ and } c \epsilon L$.

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2. L is said to distribute arbitrary joins if $a(lub \times) = lub\,\{ax : x \epsilon \times\}$ for any $a\ \epsilon L\ and\ \emptyset \neq \times \subseteq L$, in the sense that, if $lub\ \times$ exists in L, then $lub\ \{a\ x{:}x\ \epsilon\ \times\}$ exists in L and both sides of the above equation are equal. In this case, L is said to satisfy the infinite meet distributivity.
3. L is said to be distributive over if a (b c) = (a b) (a c) for all $a, b\ and\ c \epsilon L$.
4. L is said to distribute arbitrary meets if $a(glb\ \times) = glb\{a\ \ x : x\ \epsilon \times\}$ for all $a\ \epsilon\ L\ and\ \emptyset \neq \times \subseteq L$, in the sense that, when $gib\ \times$ exists in L, the $glb\{a\ x : x\ \epsilon\ \times\}$ also exists in L and both sides of the above equation are equal. In this case, L is said to satisfy the infinite join distributivity

It is well known that, in any lattice $(L,,)$, is distributive over meet if and only if

It is distributive over meet and in this case, L is said to be a distributive lattice. Also, either of (2) and (4) above implies that L is a distributive lattice. However (2) and (4) given above are not equivalent, as shown in the following examples.

Examples 1.2

1. Any totally ordered set satisfies both the infinite meet and join distributivities
2. Let N be the set of all nonnegative integers and, for any $a\ and\ b\ in\ N,$ let us define $a/b$when a divides$b$. Then $(N,/)$ is a lattice in which $a\ b = \gcd\{a, b\}\ and\ a\ \ b = lcm\{a, b\}$. This is a distributive lattice satisfying the infinite join distributivity. However, this does not satisfy the infinite meet distributivity; for, $let\ \ \times$ be the set of all odd primes. Then$2x\ = 1$ for all $x\ \epsilon \times$ and $2\ (lub \times)\ =\ 2\ 0 =\ 2\ \neq lub\{2x :\ x\epsilon \times\}$.
3. The dual of the lattice $(N,/)$ given in (2) satisfies the infinite meet distributivity and does not satisfy the infinite join distributivity.
4. Let $O(X)$ be the set of all open subsets of a topological space $\times$. Then$(O(\times), \subseteq)$ is a lattice in which, for any $[A_i]\ _{i\epsilon I}, lub\ \{A_i\}\ _{i\epsilon I}\ =\ U_{i\epsilon I}\, A_i\ and\ glb\ \{A_i\}\ _{i\epsilon I} =$ the interior of $\bigcap_{i\epsilon I} A_i$. This is a

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distributive lattice satisfying the infinite meet distributivity. However, for a general topological space X, $O(X)$ may not satisfy the infinite join distributivity.

2. HEYTING ALGEBRAS

The following is a property of lattices which is stronger than the distributivity and the infinite meet distributivity.

Definition 2.1. A lattice$(L,,)$ is said to be a Heyting algebra if, for each pair a, b of elements, there exists largest element $x\ in\ L,$, denoted by $a \to b,$ such that $x\ a\ \leq b.$ $\to$ can be treated as a binary operation on L satisfying $x\ a \leq b \Leftrightarrow x \leq a \to b$ for any $a, b\ and\ x\ in\ L.$

Examples 2.2.

1. Any totally ordered set $(L, \leq)$ with largest element 1 is a Heyting algebra in which, for any a and b in L,

$a\ b\ = \min\{a, b\},\ ab\ = \max\{a, b\}$ and

$$a \to b = \begin{cases} 1\ if a\ \leq b \\ b\ if\ b < a \end{cases}$$

2. Let $(O(X), \cap, \cup\ )$ be the lattice of all open subsets of a topological space X. This is a Heyting algebra in which, for any A and B in $O(X)$,

$A\ \to B =$ the interior of $(X - A) \cup B$

Theorem 2.3 The infinite meet distributivity holds in any Heyting algebra.

Proof: Let $(L,,)$be a Heyting algebra, $a \in L$ and X a nonempty subset of L such that $lub\ X$ exists in L. Put $y = a\ (lub\ X)$. Then $ax \leq y$ and for all $x \in X$. If z is any other upper bound of $\{a\ \ x : x \in\ X\}$,then $a\ \ x \leq z$ and hence $x\ \leq a\ \to z$ for all $x\ \ \in X$, so that $lub\ X \leq a\ \to z$ and this implies that $a(lub\ X) \leq z.$ Thus $a\ (lub\ X) = lub\ \{a\ \ x : x \in X\}.$ Therefore L satisfies the infinite meet distributivity.

The converse of the above result is not true in general. For, consider the example given below.

Example 2.4. Consider the division ordering $/$ on the set Z^+ of positive integers. Then Z^+

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is a sublattice of the lattice $(N,/)$ given in 1.2 (2) and therefore $(Z^+,/)$ is a distributive lattice without the largest element. This implies that $(Z^+,/)$ is not a Heyting algebra. However, it satisfies the infinite meet distributivity; for, $let\ a \in Z^+$ and $X \subseteq Z^+$ such that $lub\ X$ exists in Z^+. Then $lub\ X$ is a positive integer and every element of X is a divisor of $lub\ X$. This implies that X must be a finite set and we have. $a(lub\ X) = lub\ \{ax : x \in X\}$.

3. FRAMES

Recall that a partially ordered set $(L,\leq)$ is called a complete lattice if every subset of L has glb and lub in L. A complete lattice is necessarily bounded. Note that the lattice $(Z^+,/)$is not bounded above. In the following theorem, we prove that the converse of 2.3 is true for complete lattices.

Theorem 3.1. The following are equivalent to each other for any complete lattice $(L,\leq)$.

1. $(L,\leq)$ is a Heyting algebra
2. $(L,\leq)$ satisfies the infinite meet distributivity.
3. For any $a\ and\ b$ in L, the set $< a,b > = \{x \in L : ax \leq b\}$ is closed under arbitrary supremums

Proof: $(1) \Rightarrow (2)$ is proved in 2.3. For $(2) \Rightarrow (3)$, let a and b be arbitrary elements in L and $X \subseteq < a,b >$. Then $a\ x \leq b$ for all $x \in X$and $a\ (lub\ X) = lub\ \{ax : x \in X\} \leq b$ and hence $lub\ X \in < a,b >$. Thus $< a,b >$ is closed under arbitrary supremums.$(3) \Rightarrow (1)$ is clear, since $lub < a,b > \in < a,b >$ and hence $a \rightarrow b = lub < a,b >$.

Definition 3.2. A Complete Heyting Algebra is called a frame. That is, by the above theorem, a complete lattice is a frame if and only if it satisfies the infinite meet distributivity.

Examples 3.3

1. Any complete totally ordered set is a frame. In particular, the interval
[0, 1] of real numbers is a frame.
2. The lattice of open subsets of a topological space is a frame

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3. Any complete Boolean algebra is a frame, since it is a Heyting algebra in which $a \to b = a' b$, where a' is the complement of a.
4. Any finite distributive lattice is a frame.

Frames a re the most suitable candidates to have the truth values of general Fuzzy statements. For any class $\{L_i\}_{i \in I}$ of lattices, let $\pi_{i \in I} L_i$ be the product lattice in which the partial order, lattice operations, $lubs$ and $glbs$ are simply coordinatewise. The following are straight forward verifications.

Theorem 3.4. If $\{L_i\}_{i \in I}$ is a nonempty class of frames, then the product $\pi_{i \in I} L_i$ is also a frame.

Corollary 3.5. If $(L, \leq)$ is a frame and $\times$ is a nonempty set, then $L^{\times}$ is also a frame under the

pointwise ordering.

Corollary 3.6. For any positive integer n, $[0,1]^n$ is a frame.

4. ALGEBRAIC LATTICES

An element c of a complete lattice $(L, \leq)$ is called compact if, for any $\times \subseteq L$, $c \leq lub \times \Rightarrow c \leq lub\ F$ for some finite $F \subseteq \times$. If $(L, \leq)$ is a complete lattice in which every element is the lub of a set of compact elements in L, then $(L, \leq)$ is called an algebraic lattice. Equivalently, for any $a \in L$,

$a = lub\ C(a), where\ C(a) = \{c \in L : c\ is\ compact, c \leq a\}$.

Note that, for any elements a and b of an algebraic lattice, $a \leq b$ if and only if $C(a) \subseteq C(b)$.

Theorem 4.1. Any distributive algebraic lattice is a frame.

Proof: *Let* $(L, \leq)$ be a distributive algebraic lattice, $a \in L\ and\ \times \subseteq L$.

Clearly $lub\ \{a\ x : x \in \times\} \leq a\ (lub \times)$. To prove the other inequality, let c be a compact element in L such that $c \leq a\ (lub \times)$. Then $c \leq a\ and\ c \leq lub \times$. Since c is compact, there exists a finite subset F of X such that $c \leq lub\ F$.

Now,
$c \leq a(lub F) = lub\ \{a\ x : x \in F\} \leq lub\ \{a\ x : x \in \times\}$

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Thus $a\ (lub\ X) \leq lub\ \{a\ \ x : x \in X\}$ and therefore $(L, \leq)$ satisfies the infinite meet distributivity. Since $(L, \leq)$ is an algebraic lattice, it is complete also. Thus $(L, \leq)$ is a frame.

Note that, even though a frame is necessarily a distributive lattice, it may not be algebraic. For, consider the example given below.

Example 4.2.:- The interval $[0, 1]$ of real numbers is a frame and not an algebraic lattice, since 0 is the only compact element in $[0, 1]$; for, if $0 < a \leq 1$, then

$a = lub\ \{x : 0 \leq x < a$ and $a \neq lub\ F$ for any finite subset F of $[0, a)$.

In the context of [0, 1], we have an interesting characterization of totally ordered algebraic lattices.

Definition 4.3:- Let a and b be elements of a partially ordered set $(L, \leq)$ such that $a < b$. If there is no $x \in L$ such that $a < x < b$, then we say that a is covered by b and denote this by $a \prec b$.

Theorem 4.4.:- *Let* $(L, \leq)$ be a complete totally ordered set. Then $(L, \leq)$is an algebraic lattice if and only if, for any $x < y\ in\ L$, there exist $a\ and\ b\ in\ L$ such that $x \leq a \prec b \leq y$.

Proof: Suppose that $(L, \leq)$ is an algebraic lattice and $x, y \in L$ such that $x < y$. Then there exists a compact element b in L such that that $b \leq y$ and $b \nleq x$. Then $x < b \leq y$.

Put $a = lub\{z \in L : x \leq z < b\}$. Then $x \leq a \leq b \leq y$. Since b is compact, $a \neq b$. Also, for any $z \in L, a < z < b \Rightarrow x \leq z < b \Rightarrow z \leq a$, which is a contradiction. Therefore, $x \leq a \prec b \leq y$.

Conversely, suppose that the given condition is satisfied. $Let\ y \in L\ and\ x = \{c \in L : c\ is\ compact\ and\ c \leq y\}$. Then $x \leq y$. *If* $x < y$, then there exist a and b in L such that $x \leq a \prec b \leq y$. Since a is covered by b and L is totally ordered, it follows that b is compact and hence $b \leq x$, which is a contradiction. Therefore $y = x$. Thus L is an algebraic lattice.

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