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ON PRIME AND MAXIMAL SUBALGEBRAS

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Abstract-In this paper we introduce the concepts of prime subalgebras and maximal subalgebras and discuss certain important elementary properties of these. The ideals of rings or lattices can be viewed as subalgebras of some algebraic structure with the same underlying set.

Keywords- Sublanguage,Irreduciable subalgebra, augmented algebra, prime subalgebra, maximal subalgebra.

I. INTRODUCTION

The concept of fuzzy prime ideals was introduced by U.M.Swamy and K.L.N Swamy in rings and later extended to the case of lattices by U.M.Swamy and D.V.Raju and B.B.N.Koguep, C.N.Kuimi and C.Lele. we introduce the concepts of prime subalgebras and maximal subalgebras and discuss certain important elementary properties of these. These look like analogous to those of ideals and rings and lattices. if we replace the multi-plication in a ring or the meet operation in a lattice by unary operations, one for each element, then the subalgebras of the resultant algebra are pre-cisely the ideals of the given ring or lattice. In essence, the ideals of rings or lattices can be viewed as subalgebras of some algebraic structure with the same underlying set. We formalize this in the following.

De nition 1.1. Let F and F' be languages of algebras. Then F is called a sublanguage of F' or F' is called an augmented language of F if $F \subseteq F'$ and the arity map of F' restricted to F is same as the arity map of F*:*

De nition 1.2. Let F and F' be languages of algebras and F be a sublan-guage of F'*:* Then any algebra *A* of type F' can be treated as an algebra *A* of type F and, in this case, the algebra $(A; F')$ is called an augmented algebra of the algebra $(A; F)$*:*

Theorem 1.3. *Let A be an algebra of type* F *and* F' *an augmented language of* F*: Then any subalgebra of* $(A; F')$ *is a subalgebra of* $(A; F)$ *and the converse is not true.*

De nition 1.4. Let *A* be an algebra of type F*:* A subalgebra *B* of *A* is called prime if $B \neq A$ and, for any subalgebras *C* and *D* of *A;*

$$C \cap D \subseteq B \Longrightarrow C \subseteq B \text{ or } D \subseteq B$$

In other words, a prime element in the lattice of subalgebras of *A* is called a prime subalgebra of *A:* Similarly, an irreducible element in the lattice *Sub*(*A*) of subalgebras of *A* is called an irreducible subalgebra of *A*; that is, a subalgebra *B* of *A* is called irreducible if $B \neq A$ and, for any subalgebras *C* and *D* of *A;*

$$B = C \cap D \Longrightarrow B = C \text{ or } B = D:$$

Theorem 1.5. *Let A be an algebra of type* F *and the lattice of subalgebras of A be distributive. Then a subalgebra B of A is prime if and only if it is irreducible.*

The following is a characterization of the distributivity of the lattice of subalgebras of any algebra.

Theorem 1.6. Let A be an algebra of type F and Sub(A) the lattice of subalgebras of A: Then the following are equivalent to each other.

(1). (Sub(A); ⊆) is a distributive lattice

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(2). For any subalgebra B of A, B is the intersection of all prime subalge-bras of A containing B:

(3). Every subalgebra of A is the intersection of a set of prime subalgebras of A

(4). For any two distinct subalgebras B and C of A; there exist a prime subalgebra P of A containing one of B and C and not containing the other.

Proof. (1) =⇒ (2) : Suppose that (Sub(A); ⊆) is a distributive lattice. let B be a subalgebra of A and

$$C = \bigcap \{P \mid B \subseteq P \text{ and } P \text{ is a prime subalgebra of } A\}$$

Clearly $B \subseteq C$: On the other hand, let $a \in A$ such that $a \notin B$:

Consider the class

$$\mathcal{S} = \{S \in Sub(A) \mid B \subseteq S \text{ and } a \notin S\}:$$

Then $\mathcal{S} \neq \emptyset$; since $B \in \mathcal{S}$; and hence $(\mathcal{S}; \subseteq)$ is a partially ordered set. We verify that $(\mathcal{S}; \subseteq)$ satisfy the Zorn's hypothesis.

Let $\{S_i\}_{i\in I}$ be a chain in $(\mathcal{S}; \subseteq)$ and $S = \bigcup_{i\in I} S_i$:

If $f \in F$ is nullary, then $f^A \in S_i \subseteq S$ for any $i \in I$ and hence $f^A \in S$:

If $f \in F$ is n-ary, $n > 0$ and $a_1; a_2; :::; a_n \in S;$ then, for each $1 \leq j \leq n;$ there exists $i_j \in I$ such that $a_j \in S_{i_j}$: Since $\{S_i\}_{i\in I}$ is a chain, there exists $k \in I$ such that

$$a_j \in S_{i_j} \subseteq S_k \text{ for all } 1 \leq j \leq n$$

and hence $a_j \in S_k$ for all $1 \leq j \leq n:$ Since S_k is a subalgebra of $A;$ it follows that

$f^A(a_1; a_2; :::; a_n) \in S_k \subseteq S$

and hence $f^A(a_1; a_2; :::; a_n) \in S:$ Therefore, S is a subalgebra of $A:$ Since $S_i \in \mathcal{S}$ for all $i \in I;$ $a \notin S_i$ so that $a \notin S:$ Therefore $S \in \mathcal{S}$ and $S_i \subseteq S$ for all $i \in I$; that is, S is an upper bound of $\{S_i\}_{i\in I}$ in $\mathcal{S}:$ Therefore, every chain in $(\mathcal{S}; \subseteq)$ has an upper bound in $\mathcal{S}$ and hence, by the Zorn's lemma, $(\mathcal{S}; \subseteq)$ has a maximal member, say $M:$ Since $M \in \mathcal{S};$ $B \subseteq M$ and $a \notin M:$ We prove that M is a prime subalgebra of $A:$ Since $a \notin M;$ we have that $M \neq A:$ Let C and D be subalgebras of A such that $C \cap D \subseteq M:$ Suppose that $C \nsubseteq M$ and $D \nsubseteq M:$ Then $M \vee C$ and $M \vee D$ are subalgebras and each is strictly bigger than $M:$ By the maximality of $M;$ $M \vee C$ and $M \vee D$ belong to $\mathcal{S}$ and hence $a \in M \vee C$ and $a \in M \vee D:$ Now,

$$a \in (M \vee C) \cap (M \vee D) = M \vee (C \cap D)$$

which implies that $C \cap D \nsubseteq M$ (otherwise, $a \in M$) and this is a contradiction. Thus M is a prime subalgebra of A containing $B:$ This implies that $a \notin C$ (since $a \notin M$). Thus, for any $a \in A;$

$$a \notin B \Longrightarrow a \notin C$$

and hence $C \subseteq B:$ Thus $B = C:$

(2) =⇒ (3) is trivial

(3) =⇒ (4) : Let B and C be two distinct subalgebras of $A:$ Then $B \nsubseteq C$ or $C \nsubseteq B:$ With out loss of generality, we can suppose that $B \nsubseteq C:$ Choose $b \in B$ such that $b \notin C:$ By (3); there exists a prime subalgebra P of A such that $C \subseteq P$ and $b \notin P:$ Now, $b \in B - P$ and hence $B \nsubseteq P$ and $C \subseteq P:$

(4) =⇒ (1) : Let $B;$ C and D be subalgebras of $A:$ Let

$$S = B \cap (C \vee D) \text{ and } T = (B \cap C) \vee (B \cap D):$$

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It is always true that $T \subseteq S$: Suppose that $T \neq S$: Then, by (4), there exists a prime subalgebra P of A such that P contains one of S and T and does not contain the other. Since $T \subseteq S$; we should get that $T \subseteq P$ and $S * P$: Then

$$B \cap C \subseteq T \subseteq P \text{ and } B \cap D \subseteq T \subseteq P:$$

Since P is prime, either $B \subseteq P$ or ($C \subseteq P$ and $D \subseteq P$) and hence

$$B \subseteq P \text{ or } C \vee D \subseteq P:$$

Therefore $B \cap (C \vee D) \subseteq P$; that is, $S \subseteq P$; which is a contradiction. Therefore $S = T$; that is,

$$B \cap (C \vee D) = (B \cap C) \vee (B \cap D):$$

□

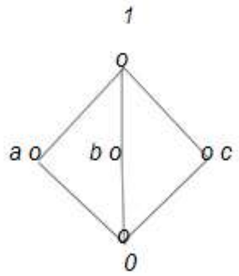
Thus $(Sub(A); \subseteq)$ is a distributive lattice.

Corollary 1.7. *Let A be an algebra of type* F*: Then every subalgebra of A is the intersection of a set of prime subalgebras of A if and only if $(Sub(A); \subseteq)$ is a frame.*

De nition 1.8. A proper subalgebra M of an algebra A is called maximal if M is not properly contained in any proper subalgebra of A:

Theorem 1.9. *Consider the lattice $(M_5; \wedge; \vee)$ where Hasse diagram is given below and let $(M_5; F')$ be the augmented algebra of M_5 as discussed in Ex-ample 2:2:5: Then the subalgebras of $(M_5; F')$ are precisely the ideals of the lattice M_5 and these are*

$$\{0\}; \{0; a\}; \{0; b\}; \{0; c\} \text{ and } \{0; a; b; c; 1\}$$



Here, each of $\{0; a\}$; $\{0; b\}$ and $\{0; c\}$ are maximal and none of these are prime; for example

$$\{0; b\} \cap \{0; c\} = \{0\} \subseteq \{0; a\}$$

$$\text{and } \{0; b\} * \{0; a\} \text{ and } \{0; c\} * \{0; a\}:$$

Remark 1.10. The lattice of subalgebras of the above algebra $(M_5; F')$ is isomorphic to the lattice $(M_5; \wedge; \vee)$ and, since M_5 is not a distributive lattice, $(Sub(M_5; F'); \subseteq)$ is also not distributive.

Theorem 1.11. *Let A be algebra of type* F *such that the lattice $(Sub(A); \subseteq)$ of subalgebras of A is distributive. Then every maximal subalgebra of A is prime.*

Proof. Let M be a maximal subalgebra of A: Then, by definition, $M \neq A$:
Let C and D be subalgebras of A such that

$$C \cap D \subseteq M \text{ and } C * M:$$

Then $M \$ M \vee C$ and, by the maximality of M; $M \vee C = A$: Now, by the distributivity in $Sub(A)$; we have

$$M = M \vee (C \cap D) = (M \vee C) \cap (M \vee D) = A \cap (M \vee D) = M \vee D$$

and hence $M = M \vee D$ and $D \subseteq M$: Therefore M is a prime subalgebra of A:

□

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