Different kinds of Fs-Set function- Images and Inverse Images

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Abstract:
In this paper, we introduce Different kinds of Fs-Set function-Images and inverse images of Fs-Subset under a given Fs-Set function.

Keywords: Fs-Set, Fs-Subset, Fs-Set function, Image, Inverse image, Complete Boolean Algebra and Complete Boolean Algebra homomorphism.

Introduction

V. Yogeswara, BiswajitRath, CH RamaSanyasi Rao, K V Uma Kameswari and D. Raghu Ram discussed in their recent research paper about Fs-set function and identified an image of an Fs-subset under an Fs-Set function and studied some properties.

Fs-set

1.1 Definition: Let U be a universal set, let $W \subseteq W_1 \subseteq U$ and W be a non-empty set. A four tuple $\mathcal{W} = (W_1, W, \overline{W}(\mu_{1W_1}, \mu_{2W}), L_W)$ is said be an Fs-set if, and only if

(1) $W \subseteq W_1$
(2) L_W is a complete Boolean Algebra
(3) $\mu_{1W_1}: W_1 \longrightarrow L_W$, $\mu_{2W}: W \longrightarrow L_W$ are such that

$$\mu_{1W_1}|W \geq \mu_{2W}$$

(4) $\overline{W}: W \longrightarrow L_W$ is defined by

$$\overline{W}x = \mu_{1W_1}x \wedge (\mu_{2W}x)^c \text{ for each } x \in W$$

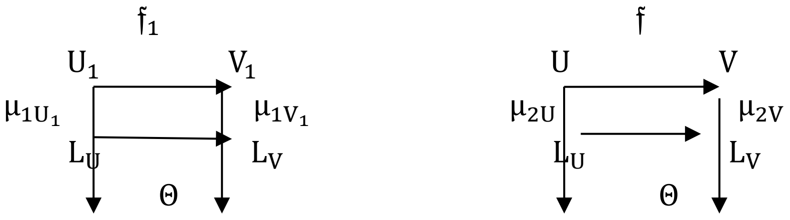
Fs-subset

1.2 Definition Suppose $\mathcal{W} = (W_1, W, \overline{W}(\mu_{1W_1}, \mu_{2W}), L_W)$ and $\mathcal{U} = (U_1, U, \overline{U}(\mu_{1U_1}, \mu_{2U}), L_U)$ are two Fs-sets. We say $\mathcal{U}$ is an Fs-subset of $\mathcal{W}$, we write $\mathcal{U} \sqsubseteq \mathcal{W}$, if, and only if

(1) $U_1 \sqsubseteq W_1$, $W \sqsubseteq U$
(2) L_U is a complete subalgebra of L_W or $L_U \leq L_W$
(3) $\mu_{1U_1} \leq \mu_{1W_1}|U_1$, and $\mu_{2U}|W \geq \mu_{2W}$

FS-SET FUNCTION

1.3 Definition For $= \left(U_1, U, \overline{U}\left(\mu_{1U_1}, \mu_{2U}\right), L_U\right)$ and $\mathcal{V} = \left(V_1, V, \overline{V}\left(\mu_{1V_1}, \mu_{2V}\right), L_V\right) \sqsubseteq \mathcal{W}$, a triplet $\bar{\mathfrak{f}} = (\mathfrak{f}_1, \mathfrak{f}, \Theta): \mathcal{U} \to \mathcal{V}$ is an Fs-set function (with the help of the following diagram) if, and only if



($\bar{\mathfrak{f}}: \mathcal{U} \to \mathcal{V}$with figure)

(a) $\mathfrak{f} = \mathfrak{f}_1\big|_U^V : U \longrightarrow V$ is surjective

Here $\mathfrak{f}_1\big|_U^V = \mathfrak{f}_1 \cap (U \times V)$

(b) $\Theta : L_U \longrightarrow L_V$ is Complete homomorphism

THE IMAGES OF FS-SUBSETS

1.4 Definition Suppose$\mathcal{U} = \left(U_1, U, \overline{U}\left(\mu_{1U_1}, \mu_{2U}\right), L_U\right), \mathcal{V} = \left(V_1, V, \overline{V}\left(\mu_{1V_1}, \mu_{2V}\right), L_V\right)$,

$\mathcal{P} = \left(P_1, P, \overline{P}\left(\mu_{1P_1}, \mu_{2P}\right), L_P\right), \mathcal{P} \sqsubseteq \mathcal{U}and\bar{\mathfrak{f}}: \mathcal{U} \to \mathcal{V}$ be an Fs-set function where $P = U$ and

$\mathfrak{f} = \mathfrak{f}_1\big|_U^V : U \longrightarrow V$be onto.

We define by $\bar{\mathfrak{f}}(\mathcal{P})$ – the image of $\mathcal{P}$ under $\bar{\mathfrak{f}}$, we define it as

$\bar{\mathfrak{f}}(\mathcal{P}) = \mathcal{Q} = \left(Q_1, Q, \overline{Q}\left(\mu_{1Q_1}, \mu_{2Q}\right) L_Q\right)$, where

(1) $Q_1 = \mathfrak{f}_1(P_1)$

(2) $Q = \mathfrak{f}(P) = \mathfrak{f}(U) = V$

(3) $\mu_{1Q_1} : Q_1 \longrightarrow L_V$ is defined by

$$\mu_{1Q_1}y = \begin{cases} \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1x \\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right)\right], \text{if } y \in V \\ \mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1x \\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right), \text{if } y \notin V \end{cases}$$

(4) $\mu_{2Q} : Q \longrightarrow L_V$ is defined by

$$\mu_{2Q}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}x \\ x\in P}}{\curlyvee} \Theta\mu_{2P}x\right)\right]$$

(5) $L_Q = \left(\left[\mu_{1Q_1}(Q_1)\right]\right)$ = The complete subalgebra generated by $\left[\mu_{1Q_1}(Q_1)\right]$, where $\left[\mu_{1Q_1}(Q_1)\right] =$ The complete ideal generated by $\mu_{1Q_1}(Q_1)$ in L_V

1.5 Complete Boolean Algebra A Boolean algebra L is said to be a complete Boolean algebra if , and only if

a. sup(W) and inf(W) exist in L for any $W \subseteq L$

b. $a \wedge (\vee_{i\in I} b_i) = \vee_{i\in I}(a \wedge b_i)$ for any $\{b_i\}_{i\in I} \subseteq L$

1.6 Complete Boolean Algebra homomorphism Let W and U be a pair of complete Boolean algebras and $V \sqsubseteq W$. A Boolean algebra homomorphism $f: W \to U$ is said to a complete Boolean algebra homomorphism if, and only, if

a. $f(\mathrm{Inf}(V)) = \inf(f(V))$
b. $f(\sup(V)) = \sup(f(V))$

Section-2

2.1 Definition

FS-SUBSET -INVERSE IMAGE

Suppose $\mathcal{P} \sqsubseteq \mathcal{U}$, $\bar{\mathfrak{f}}: \mathcal{U} \longrightarrow \mathcal{V}$ is an Fs- set function and $\mathfrak{f} = \mathfrak{f}_1|_U^V: U \longrightarrow V$ be onto.

Let $\mathcal{Q} = (Q_1, Q, \overline{Q}(\mu_{1Q_1}, \mu_{2Q}), L_Q) \sqsubseteq \mathcal{V}$ be such that Q=V. We define $\bar{\mathfrak{f}}^{-1}(\mathcal{Q})$ –the inverse image $\mathcal{Q}$ as $\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \mathcal{P} = (P_1, P, \overline{P}(\mu_{1P_1}, \mu_{2P}), L_P)$, where

a) $P_1 = \mathfrak{f}_1^{-1}(Q_1)$
b) $P = \mathfrak{f}^{-1}(Q) = \mathfrak{f}^{-1}(V) = U$
c) $\mu_{1P_1}: P_1 \longrightarrow L_P$ is defined by

$$\mu_{1P_1}x = \begin{cases} \mu_{1U_1}x & ,\text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right)\right] & ,x \in U \\ \mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right) & ,x \notin U \end{cases}$$

d) $\mu_{2P}: P \longrightarrow L_P$ is defined by

$$\mu_{2P}x = \begin{cases} \mu_{2U}x & ,\text{whenever } \Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right] & \end{cases}$$

e) $L_P = L_U$

2.1.1 Theorem: Suppose $\bar{\mathfrak{f}}_d: \mathcal{U} \longrightarrow \mathcal{V}$ with $\mathcal{Q} \sqsubseteq \mathcal{V}$, $Q = V$. Then $\bar{\mathfrak{f}}\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) \sqsubseteq \mathcal{Q}$.

Proof: We have, $\Theta \circ \mu_{2U}x \le \mu_{2V}y \le \mu_{2Q}y \le \mu_{1Q_1}y \le \mu_{1V_1}y \le \Theta \circ \mu_{1U_1}x$ for $x \in U$

We have
$$\mathcal{U} = (U_1, U, \overline{U}(\mu_{1U_1}, \mu_{2U}), L_U),$$
$$\mathcal{V} = (V_1, V, \overline{V}(\mu_{1V_1}, \mu_{2V}), L_V) \text{ and}$$
$$\mathcal{Q} = (Q_1, Q, \overline{Q}(\mu_{1Q_1}, \mu_{2Q}), L_Q)$$

Let $\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \mathcal{P} = (P_1, P, \overline{P}(\mu_{1P_1}, \mu_{2P}), L_P)$, Where

(a) $P_1 = \mathfrak{f}_1^{-1}(Q_1)$
(b) $P = \mathfrak{f}^{-1}(Q)$
(c) $\mu_{1P_1}: P_1 \longrightarrow L_P$ is defined by

$$\mu_{1P_1}x = \begin{cases} \mu_{2U}x & ,\text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x\\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right)\right] & ,x \in U \\ \mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x\\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right) & ,x \notin U \end{cases}$$

(d) $\mu_{2P}: P \longrightarrow L_P$ is defined by

$$\mu_{2P}x = \begin{cases} \mu_{2U}x & ,\text{whenever } \Theta^{-1}\mu_{2Q}\mathfrak{f}x = \Theta \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x\\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right] & \end{cases}$$

(e) $L_P = L_U$

Again suppose $\bar{\mathfrak{f}}\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \bar{\mathfrak{f}}(\mathcal{P}) = \mathcal{R} = \left(R_1, R, \bar{R}(\mu_{1R_1}, \mu_{2R}), L_R\right)$, Where

(f) $R_1 = \mathfrak{f}_1(P_1) = \mathfrak{f}_1\left(\mathfrak{f}_1^{-1}(Q_1)\right) \sqsubseteq Q_1$ $(\because \mathfrak{f}_1$ is onto)

(g) $R = \mathfrak{f}(P) = \mathfrak{f}\left(\mathfrak{f}^{-1}(Q)\right) = \mathfrak{f}\left(\mathfrak{f}^{-1}(V)\right) = V = Q$

(h) $\mu_{1R_1}: R_1 \longrightarrow L_V$ is defined by

$$\mu_{1R_1}y = \begin{cases} \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right)\right] & ,y \in V \\ \mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right) & ,y \notin V \end{cases}$$

(i) $\mu_{2R}: R \longrightarrow L_V$ is defined by

$$\mu_{2R}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}x\\ x\in P}}{\curlyvee} \Theta\mu_{2P}x\right)\right]$$

(j) $L_R = \left(\left[\mu_{1R_1}(R_1)\right]\right)$, (See Chapter-III, Remark 3.0)

Now $\mathcal{R} \sqsubseteq \mathcal{Q}$ is needed i.e. to show,

(k) $R_1 \sqsubseteq Q_1, R \sqsupseteq Q$

(l) $L_R \leq L_Q$

(m) $\mu_{1R_1}y \leq \mu_{1Q_1}y, \mu_{2R}y \geq \mu_{2Q}y$

(f) and (g) $\Longrightarrow$(k)

(e) and (j)$\Longrightarrow$(l)

(m): For $x \in U$,

$$\mu_{1R_1}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right)\right]$$

$$= \mu_{2V}\mathcal{y} \curlyvee \left[\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \Theta\left(\mu_{2U}\mathcal{x} \curlyvee \left[\mu_{1U_1}\mathcal{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]\right)\right)\right]$$

$$= \mu_{2V}\mathcal{y} \curlyvee \left[\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\Theta\mu_{2U}\mathcal{x} \curlyvee \left[\Theta\mu_{1U_1}\mathcal{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]\right)\right)\right]$$

Case: 1a: $\Theta^{-1}\mu_{1Q_1}\mathcal{y} = \varphi$, $\mu_{1R_1}\mathcal{y} = \mu_{1Q_1}\mathcal{y}$

Case: 1b: $\Theta^{-1}\mu_{1Q_1}\mathcal{y} \neq \varphi$

$$\mu_{1R_1}\mathcal{y} = \mu_{2V}\mathcal{y} \curlyvee \left[\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\Theta\mu_{2U}\mathcal{x} \curlyvee \left[\Theta\mu_{1U_1}\mathcal{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]\right)\right)\right]$$

$$= \mu_{2V}\mathcal{y} \curlyvee \left[\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\Theta\mu_{2U}\mathcal{x} \curlyvee \left[\Theta\mu_{1U_1}\mathcal{x} \curlywedge \mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right]\right)\right)\right]$$

$$= \mu_{2V}\mathcal{y} \curlyvee \left[\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\left(\Theta\mu_{2U}\mathcal{x} \curlyvee \mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right)\right)\right]$$

$$= \mu_{2V}\mathcal{y} \curlyvee \left[\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]$$

$= \mu_{2V}\mathcal{y} \curlyvee \left(\mu_{1V_1}\mathcal{y} \curlywedge \mu_{1Q_1}\mathcal{y}\right) = \mu_{2V}\mathcal{y} \curlyvee \mu_{1Q_1}\mathcal{y} = \mu_{1Q_1}\mathcal{y} = \mu_{1Q_1\mathfrak{f}_1\mathcal{x}}$, whenever $\mathcal{x} \in U$

For $\mathcal{y} \notin V$,

$$\mu_{1R_1}\mathcal{y} = \mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \Theta\mu_{1P_1}\mathcal{x}\right)$$

$$= \mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \Theta\left(\mu_{2U}\mathcal{x} \curlyvee \left[\mu_{1U_1}\mathcal{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]\right)\right)$$

$$= \mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\Theta\mu_{2U}\mathcal{x} \curlyvee \left[\Theta\mu_{1U_1}\mathcal{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]\right)\right)$$

Case: 2a: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x} = \Theta$, $\mu_{1R_1}\mathfrak{f}_1\mathcal{x} = \mu_{1Q_1}\mathfrak{f}_1\mathcal{x}$

Case: 2b: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x} \neq \varphi$

$$\mu_{1R_1}\mathcal{y} = \mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\Theta\mu_{2U}\mathcal{x} \curlyvee \left[\Theta\mu_{1U_1}\mathcal{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathcal{x}\right)\right]\right)\right)$$

$$= \mu_{1V_1}\mathcal{y} \curlywedge \left(\mu_{1V_1}\mathcal{y} \curlywedge \left(\curlyvee_{\substack{\mathcal{y}=\mathfrak{f}_1\mathcal{x}\\ \mathcal{x}\in P_1}} \left(\left(\Theta\mu_{2U}\mathcal{x} \curlyvee \left[\Theta\mu_{1U_1}\mathcal{x} \curlywedge \mu_{1Q_1}\mathcal{y}\right]\right)\right)\right)\right)$$

$$= \mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x} \\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \vee \mu_{1Q_1}\mathscr{y}\right) \right)$$

$$= \mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x} \\ \mathscr{x}\in P_1}} \mu_{1Q_1}\mathscr{y} \right) = \mu_{1V_1}\mathscr{y} \wedge \mu_{1Q_1}\mathscr{y} = \mu_{1Q_1}\mathscr{y}$$

$$\mu_{2R}\mathscr{y} = \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}\mathscr{x} \\ \mathscr{x}\in P}} \Theta\mu_{2P}\mathscr{x} \right) \right]$$

$$= \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}\mathscr{x} \\ \mathscr{x}\in P}} \Theta \left\{ \mu_{2U}\mathscr{x} \vee \left[\mu_{1U_1}\mathscr{x} \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}\mathscr{x} \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}\mathscr{x} \right) \right] \right\} \right) \right]$$

$$= \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}\mathscr{x} \\ \mathscr{x}\in P}} \left\{ \Theta\mu_{2U}\mathscr{x} \vee \left[\Theta\mu_{1U_1}\mathscr{x} \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}\mathscr{x} \\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{2Q}\mathfrak{f}\mathscr{x} \right) \right] \right\} \right) \right]$$

Case: 3a: $\Theta^{-1}\mu_{2Q}\mathfrak{f}\mathscr{x} = \Theta, \mu_{2R}\mathfrak{f}\mathscr{x} = \mu_{2Q}\mathfrak{f}\mathscr{x}$

Case: 3b: $\Theta^{-1}\mu_{2Q}\mathfrak{f}\mathscr{x} \neq \varphi$

$$\mu_{2R}\mathscr{y} = \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}\mathscr{x} \\ \mathscr{x}\in P}} \left\{ \Theta\mu_{2U}\mathscr{x} \vee \left[\Theta\mu_{1U_1}\mathscr{x} \wedge \mu_{2Q}\mathscr{y} \right] \right\} \right) \right]$$

$$= \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}\mathscr{x} \\ \mathscr{x}\in P}} \left\{ \Theta\mu_{2U}\mathscr{x} \vee \mu_{2Q}\mathscr{y} \right\} \right) \right]$$

$$= \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \left(\bigvee_{\substack{\mathscr{y}=\mathfrak{f}\mathscr{x} \\ \mathscr{x}\in P}} \mu_{2Q}\mathscr{y} \right) \right] = \mu_{2V}\mathscr{y} \vee \left[\mu_{1V_1}\mathscr{y} \wedge \mu_{2Q}\mathscr{y} \right] = \mu_{2V}\mathscr{y} \vee \mu_{2Q}\mathscr{y} = \mu_{2Q}\mathscr{y}$$

Hence $\mathcal{R} \sqsubseteq \mathcal{Q}$ i.e.,

$$\bar{\mathfrak{f}}\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) \sqsubseteq \mathcal{Q}.$$

2.1.2 Corollary: Suppose $\bar{\mathfrak{f}}_d: \mathcal{U} \longrightarrow \mathcal{V}$, $\mathfrak{f}_1$ is onto, $\mathcal{Q} \sqsubseteq \mathcal{V}, Q = V$. Then $\bar{\mathfrak{f}}\bar{\mathfrak{f}}^{-1}(\mathcal{Q})$ and $\mathcal{Q}$ are full − equal.

Proof: We have, $\Theta \circ \mu_{2U}\mathscr{x} \leq \mu_{2V}\mathscr{y} \leq \mu_{2Q}\mathscr{y} \leq \mu_{1Q_1}\mathscr{y} \leq \mu_{1V_1}\mathscr{y} \leq \Theta \circ \mu_{1U_1}\mathscr{x}$ for $\mathscr{x} \in U$

We have $\mathcal{U} = \left(U_1, U, \bar{U}\left(\mu_{1U_1}, \mu_{2U}\right), L_U\right)$

$\mathcal{V} = \left(V_1, V, \bar{V}\left(\mu_{1V_1}, \mu_{2V}\right), L_V\right)$ and

$$\mathcal{Q} = \left(Q_1, Q, \bar{Q}\left(\mu_{1Q_1}, \mu_{2Q}\right), L_Q\right)$$

Let, $\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \mathcal{P} = \left(P_1, P, \bar{P}\left(\mu_{1P_1}, \mu_{2P}\right), L_P\right)$, where

(a) $P_1 = \mathfrak{f}_1^{-1}(Q_1)$

(b) $P = \mathfrak{f}^{-1}(Q)$

(c) $\mu_{1P_1}: P_1 \longrightarrow L_P$ is defined by

$$\mu_{1P_1}x = \begin{cases} \mu_{1U_1}x & ,\text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x \right)\right] & ,x \in U \\ \mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x \right) & ,x \notin U \end{cases}$$

(d) $\mu_{2P}: P \longrightarrow L_P$ is defined by

$$\mu_{2P}x = \begin{cases} \mu_{2U}x & ,\text{whenever } \Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}}{\curlyvee} \Theta^{-1}\mu_{2Q}\mathfrak{f}x \right)\right] \end{cases}$$

(e) $L_P = L_U$

Again suppose $\bar{\mathfrak{f}}\,\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \bar{\mathfrak{f}}(\mathcal{P}) = \mathcal{R} = (R_1, R, \bar{R}(\mu_{1R_1}, \mu_{2R}), L_R)$, where

(f) $R_1 = \mathfrak{f}_1(P_1) = \mathfrak{f}_1(\mathfrak{f}_1^{-1}(Q_1)) = Q_1$

(g) $R = \mathfrak{f}(P) = \mathfrak{f}(\mathfrak{f}^{-1}(Q)) = \mathfrak{f}(\mathfrak{f}^{-1}(V)) = V = Q$

(h) $\mu_{1R_1}: R_1 \longrightarrow L_V$ is defined by

$$\mu_{1R_1}y = \begin{cases} \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x \right)\right] & ,y \in V \\ \mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x \right) & ,y \notin V \end{cases}$$

(i) $\mu_{2R}: R \longrightarrow L_V$ is defined by

$$\mu_{2R}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}x \\ x\in P}}{\curlyvee} \Theta\mu_{2P}x \right)\right]$$

(j) $L_R = ([\mu_{1R_1}(R_1)])$ (See Chapter-III, Remark-3.0)

Now $\mathcal{R}$ and $\mathcal{Q}$ are full-equal needed i.e. to show,

(k) $R_1 = Q_1$

(l) $R = Q$

(m) $L_R = L_Q$

(n) $\mu_{1R_1}y = \mu_{1Q_1}y$

(o) $\mu_{2R}y = \mu_{2Q}y$

(f) and (g) $\Longrightarrow$ (k) and (l)

(e) and (j) $\Longrightarrow$(m)

(n): For $x \in U$, $$\mu_{1R_1}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}}{\curlyvee} \Theta\mu_{1P_1}x \right)\right]$$

$$= \mu_{2V}\mathscr{y} \curlyvee \left[\mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \Theta\left(\mu_{2U}\mathscr{x} \curlyvee \left[\mu_{1U_1}\mathscr{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]\right)\right)\right]$$

$$= \mu_{2V}\mathscr{y} \curlyvee \left[\mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \left[\Theta\mu_{1U_1}\mathscr{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]\right)\right)\right]$$

Case: 1a: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x} = \varphi$, $\mu_{1R_1}\mathfrak{f}_1\mathscr{x} = \mu_{1Q_1}\mathfrak{f}_1\mathscr{x}$

Case: 1b: :$\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x} \neq \varphi$

$$\mu_{1R_1}\mathscr{y} = \mu_{2V}\mathscr{y} \curlyvee \left[\mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \left[\Theta\mu_{1U_1}\mathscr{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]\right)\right)\right]$$

$$= \mu_{2V}\mathscr{y} \curlyvee \left[\mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \left[\Theta\mu_{1U_1}\mathscr{x} \curlywedge \mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right]\right)\right)\right]$$

$$= \mu_{2V}\mathscr{y} \curlyvee \left[\mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right)\right]$$

$$= \mu_{2V}\mathscr{y} \curlyvee \left[\mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]$$

$= \mu_{2V}\mathscr{y} \curlyvee \left(\mu_{1V_1}\mathscr{y} \curlywedge \mu_{1Q_1}\mathscr{y}\right) = \mu_{2V}\mathscr{y} \curlyvee \mu_{1Q_1}\mathscr{y} = \mu_{1Q_1}\mathscr{y}$,whenever $\mathscr{x} \in U$

For $\mathscr{y} \notin V$,

$$\mu_{1R_1}\mathscr{y} = \mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \Theta\mu_{1P_1}\mathscr{x}\right)$$

$$= \mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \Theta\left(\mu_{2U}\mathscr{x} \curlyvee \left[\mu_{1U_1}\mathscr{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]\right)\right)$$

$$= \mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \left[\Theta\mu_{1U_1}\mathscr{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]\right)\right)$$

Case: 2a: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x} = \varphi$,$\mu_{1R_1}\mathfrak{f}_1\mathscr{x} = \mu_{1Q_1}\mathfrak{f}_1\mathscr{x}$

Case: 2b: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x} \neq \varphi$

$$\mu_{1R_1}\mathscr{y} = \mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \left[\Theta\mu_{1U_1}\mathscr{x} \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1\mathscr{x}\right)\right]\right)\right)$$

$$= \mu_{1V_1}\mathscr{y} \curlywedge \left(\curlyvee_{\substack{\mathscr{y}=\mathfrak{f}_1\mathscr{x}\\ \mathscr{x}\in P_1}} \left(\Theta\mu_{2U}\mathscr{x} \curlyvee \left[\Theta\mu_{1U_1}\mathscr{x} \curlywedge \mu_{1Q_1}\mathscr{y}\right]\right)\right)$$

$$= \mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}_1\mathcal{X} \\ \mathcal{X}\in P_1}} (\Theta\mu_{2U}\mathcal{X} \vee \mu_{1Q_1}\mathcal{Y}) \right)$$

$$= \mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}_1\mathcal{X} \\ \mathcal{X}\in P_1}} \mu_{1Q_1}\mathcal{Y} \right) = \mu_{1V_1}\mathcal{Y} \wedge \mu_{1Q_1}\mathcal{Y} = \mu_{1Q_1}\mathcal{Y}$$

$$\mu_{2R}\mathcal{Y} = \mu_{2V}\mathcal{Y} \vee \left[\mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}\mathcal{X} \\ \mathcal{X}\in P}} \Theta\mu_{2P}\mathcal{X} \right) \right]$$

$$= \mu_{2V}\mathcal{Y} \vee \left[\mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}\mathcal{X} \\ \mathcal{X}\in P}} \Theta \left\{ \mu_{2U}\mathcal{X} \vee \left[\mu_{1U_1}\mathcal{X} \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}\mathcal{X} \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}\mathcal{X} \right) \right] \right\} \right) \right]$$

$$= \mu_{2V}\mathcal{Y} \vee \left[\mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}\mathcal{X} \\ \mathcal{X}\in P}} \left\{ \Theta\mu_{2U}\mathcal{X} \vee \left[\Theta\mu_{1U_1}\mathcal{X} \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}\mathcal{X} \\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{2Q}\mathfrak{f}\mathcal{X} \right) \right] \right\} \right) \right]$$

Case: 3a: $\Theta^{-1}\mu_{2Q}\mathcal{Y} = \Theta, \mu_{2R}\mathcal{Y} = \mu_{2Q}\mathcal{Y}$

Case: 3b: $\Theta^{-1}\mu_{2Q}\mathfrak{f}\mathcal{X} \neq \varphi$

$$\mu_{2R}\mathcal{Y} = \mu_{2V}\mathcal{Y} \vee \left[\mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}\mathcal{X} \\ \mathcal{X}\in P}} \{\Theta\mu_{2U}\mathcal{X} \vee [\Theta\mu_{1U_1}\mathcal{X} \wedge \mu_{2Q}\mathcal{Y}]\} \right) \right]$$

$$= \mu_{2V}\mathcal{Y} \vee \left[\mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}\mathcal{X} \\ \mathcal{X}\in P}} \{\Theta\mu_{2U}\mathcal{X} \vee \mu_{2Q}\mathcal{Y}\} \right) \right]$$

$$= \mu_{2V}\mathcal{Y} \vee \left[\mu_{1V_1}\mathcal{Y} \wedge \left(\bigvee_{\substack{\mathcal{Y}=\mathfrak{f}\mathcal{X} \\ \mathcal{X}\in P}} \mu_{2Q}\mathcal{Y} \right) \right] = \mu_{2V}\mathcal{Y} \vee [\mu_{1V_1}\mathcal{Y} \wedge \mu_{2Q}\mathcal{Y}] = \mu_{2V}\mathcal{Y} \vee \mu_{2Q}\mathcal{Y} = \mu_{2Q}\mathcal{Y}$$

Hence $\mathcal{R} = \mathcal{Q}$ i.e., the full-equality of $\bar{\mathfrak{f}}\,\bar{\mathfrak{f}}^{-1}(\mathcal{Q})$ and $\mathcal{Q}$ follows

2.2 Theorem: Suppose$\bar{\mathfrak{f}}_p : \mathcal{U} \longrightarrow \mathcal{V}$ with$\mathcal{Q} \sqsubseteq \mathcal{V}$, Q = V. Then $\bar{\mathfrak{f}}\,\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) \sqsubseteq \mathcal{Q}$.

Proof: We have, $\Theta \circ \mu_{2U}\mathcal{X} = \mu_{2V}\mathcal{Y} = \mu_{2Q}\mathcal{Y} = \mu_{1Q_1}\mathcal{Y} = \mu_{1C_1}\mathcal{Y} = \Theta \circ \mu_{1U_1}\mathcal{X}$ for $\mathcal{X} \in U$

$$\mathcal{U} = \left(U_1, U, \bar{U}(\mu_{1U_1}, \mu_{2U}), L_U\right)$$

$$\mathcal{V} = \left(V_1, V, \bar{V}(\mu_{1V_1}, \mu_{2V}), L_V\right) \text{and}$$

$$\mathcal{Q} = \left(Q_1, Q, \bar{Q}(\mu_{1Q_1}, \mu_{2Q}), L_Q\right)$$

Let $\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \mathcal{P} = \left(P_1, P, \bar{P}(\mu_{1P_1}, \mu_{2P}), L_P\right)$,Where

(a) $P_1 = \mathfrak{f}_1^{-1}(Q_1)$

(b) $P = \mathfrak{f}^{-1}(Q)$

(c) $\mu_{1P_1} : P_1 \longrightarrow L_P$ is defined by

$$\mu_{1P_1}x = \begin{cases} \mu_{1U_1}x & ,\text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x \\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right] & ,x \in U \\ \mu_{1U_1}x \curlywedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x \\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right) & ,x \notin U \end{cases}$$

(d) $\mu_{2P}: P \longrightarrow L_P$ is defined by

$$\mu_{2P}x = \begin{cases} \mu_{2U}x & ,\text{whenever } \Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right] & \end{cases}$$

(e) $L_P = L_U$

Again suppose $\bar{\mathfrak{f}}\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \bar{\mathfrak{f}}(\mathcal{P}) = \mathcal{R} = \left(R_1, R, \overline{R}\left(\mu_{1R_1}, \mu_{2R}\right), L_R\right)$,
where

(f) $R_1 = \mathfrak{f}_1(P_1) = \mathfrak{f}_1\left(\mathfrak{f}_1^{-1}(Q_1)\right) \sqsubseteq Q_1$

(g) $R = \mathfrak{f}(P) = \mathfrak{f}\left(\mathfrak{f}^{-1}(Q)\right) = \mathfrak{f}\left(\mathfrak{f}^{-1}(V)\right) = V = Q$

(h) $\mu_{1R_1}: R_1 \longrightarrow L_V$ is defined by

$$\mu_{1R_1}y = \begin{cases} \mu_{1Q_1}\mathfrak{f}_1x & ,\text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x = \varphi \\ \mu_{2V}\mathfrak{f}_1x \curlyvee \left[\mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\bigvee_{\substack{y=\mathfrak{f}_1x \\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right] & ,y \in V \\ \mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\bigvee_{\substack{y=\mathfrak{f}_1x \\ x\in P_1}} \Theta\mu_{1P_1}x\right) & ,y \notin V \end{cases}$$

(i) $\mu_{2R}: R \longrightarrow L_V$ is defined by

$$\mu_{2R}y = \begin{cases} \mu_{2Q}\mathfrak{f}_1x & ,\text{whenever } \Theta^{-1\mu_{2Q}}\mathfrak{f}x = \varphi \\ \mu_{2V}\mathfrak{f}_1x \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\mu_{2P}x\right)\right] & \end{cases}$$

(j) $L_R = \left(\left[\mu_{1R_1}(R_1)\right]\right)$, where $\left[\mu_{1R_1}(R_1)\right]$ is the complete ideal generated by $\mu_{1R_1}(R_1)$ and $\left(\left[\mu_{1R_1}(R_1)\right]\right)$ is the complete subalgebra generated by $\left[\mu_{1R_1}(R_1)\right]$

$\mathcal{R} \sqsubseteq \mathcal{Q}$is needed i.e. it is sufficient to show that,

(k) $R_1 \sqsubseteq Q_1, R \sqsupseteq Q$

(l) $L_R \leq L_Q$

(m) $\mu_{1R_1}y \leq \mu_{1Q_1}y, \mu_{2R}y \geq \mu_{2Q}y$

(f) and (g)$\Longrightarrow$ (k)

(e) and (j)$\Longrightarrow$(l)

(m): For $x \in U$,

$$\mu_{1R_1}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]$$

$$= \mu_{2V}\mathfrak{f}_1x \curlyvee \left[\mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\left(\mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right]\right)\right)\right]$$

$$= \mu_{2V}\mathfrak{f}_1x \curlyvee \left[\mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \left(\Theta\mu_{2U}x \curlyvee \left[\Theta\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right]\right)\right)\right]$$

Case: 1a: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x = \varphi$, $\mu_{1R_1}\mathfrak{f}_1x = \mu_{1Q_1}\mathfrak{f}_1x$

Case: 1b: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x \neq \varphi$

$$\mu_{1R_1}\mathfrak{f}_1x = \mu_{2V}\mathfrak{f}_1x \curlyvee \left[\mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \left(\Theta\mu_{2U}x \curlyvee \left[\Theta\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right]\right)\right)\right]$$

$$= \mu_{2V}\mathfrak{f}_1x \curlyvee \left[\mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \left(\Theta\mu_{2U}x \curlyvee \left[\Theta\mu_{1U_1}x \curlywedge \mu_{1Q_1}\mathfrak{f}_1x\right]\right)\right)\right]$$

$$= \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \left(\Theta\mu_{2U}x \curlyvee \mu_{1Q_1}\mathfrak{f}_1x\right)\right)\right]$$

$$= \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \mu_{1Q_1}\mathfrak{f}_1x\right)\right]$$

$= \mu_{2V}y \curlyvee \left(\mu_{1V_1}y \curlywedge \mu_{1Q_1}y\right) = \mu_{2V}y \curlyvee \mu_{1Q_1}y = \mu_{1Q_1}y$,whenever $x \in U$

For $y \notin V$,

$$\mu_{1R_1}y = \mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)$$

$$= \mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\left(\mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right]\right)\right)$$

$$= \mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \left(\Theta\mu_{2U}x \curlyvee \left[\Theta\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right]\right)\right)$$

Case: 2a: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x = \varphi$,$\mu_{1R_1}\mathfrak{f}_1x = \mu_{1Q_1}\mathfrak{f}_1x$

Case: 2b: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x \neq \varphi$

$$\mu_{1R_1}\mathfrak{f}_1x = \mu_{1V_1}\mathfrak{f}_1x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \left(\Theta\mu_{2U}x \curlyvee \left[\Theta\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right]\right)\right)$$

$$= \mu_{1V_1}\mathfrak{f}_1 x \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \mu_{1Q_1}\mathfrak{f}_1 x\right]\right)\right)$$

$$= \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \mu_{1Q_1}\mathfrak{f}_1 x\right)\right)$$

$$= \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}} \mu_{1Q_1}\mathfrak{f}_1 x\right)$$

$= \mu_{1V_1}y \wedge \mu_{1Q_1}y = \mu_{1Q_1}y$,whenever $x \in U$

$= \mu_{1Q_1}\mathfrak{f}_1 x = \mu_{1Q_1}y$

$$\mu_{2R}y = \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\mu_{2P}x\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\left\{\mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right]\right\}\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \left\{\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right]\right\}\right)\right]$$

Case: 3a: $\Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi, \mu_{2R}\mathfrak{f}x = \mu_{2Q}\mathfrak{f}x$

Case: 3b: $\Theta^{-1}\mu_{2Q}\mathfrak{f}x \neq \varphi$

$$\mu_{2R}y = \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \left\{\left\{\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \mu_{2Q}\mathfrak{f}x\right]\right\}\right\}\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \left\{\Theta\mu_{2U}x \vee \mu_{2Q}\mathfrak{f}x\right\}\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \mu_{2Q}\mathfrak{f}x\right)\right] = \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \mu_{2Q}y\right] = \mu_{2V}y \vee \mu_{2Q}y$$

$$= \mu_{2Q}\mathfrak{f}x = \mu_{2Q}y$$

Hence $\mathcal{R} \sqsubseteq \mathcal{Q}$ i.e.,

$$\bar{\mathfrak{f}}\,\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) \sqsubseteq \mathcal{Q}.$$

2.2.1 Corollary

Suppose $\bar{\mathfrak{f}}_p: \mathcal{U} \longrightarrow \mathcal{V}$ with $\mathfrak{f}_1$ is onto, $\mathcal{Q} \sqsubseteq \mathcal{V}$, $Q = V$. Then $\bar{\mathfrak{f}}\,\bar{\mathfrak{f}}^{-1}(\mathcal{Q})$ and $\mathcal{Q}$ are full-equal.

Proof:

We have, $\Theta \circ \mu_{2U}x = \mu_{2V}y = \mu_{2Q}y = \mu_{1Q_1}y = \mu_{1V_1}y = \Theta \circ \mu_{1U_1}x$ for $x \in U$

$$\mathcal{U} = \left(U_1, U, \bar{U}\left(\mu_{1U_1,}\mu_{2U}\right), L_U\right)$$

$$\mathcal{V} = \left(V_1, V, \bar{V}\left(\mu_{1V_1,}\mu_{2V}\right), L_V\right) \text{and}$$

$$\mathcal{Q} = \left(Q_1, Q, \overline{Q}(\mu_{1Q_1}, \mu_{2Q}), L_Q\right)$$

Let, $\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \mathcal{P} = \left(P_1, P, \overline{P}(\mu_{1P_1}, \mu_{2P}), L_P\right)$, where

(a) $P_1 = \mathfrak{f}_1^{-1}(Q_1)$

(b) $P = \mathfrak{f}^{-1}(Q)$

(c) $\mu_{1P_1}: P_1 \longrightarrow L_P$ is defined by

$$\mu_{1P_1}x = \begin{cases} \mu_{1U_1}x & , \text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha = \mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha \in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right)\right] & , x \in U \\ \mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha = \mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha \in L_U}}{\curlyvee} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right) & , x \notin U \end{cases}$$

(d) $\mu_{2P}: P \longrightarrow L_P$ is defined by

$$\mu_{2P}x = \begin{cases} \mu_{2U}x & , \text{whenever } \Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\underset{\substack{\Theta\alpha = \mu_{2Q}\mathfrak{f}x \\ \alpha \in L_U}}{\curlyvee} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right] \end{cases}$$

(e) $L_P = L_U$

Again suppose $\bar{\mathfrak{f}}\,\bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \bar{\mathfrak{f}}(\mathcal{P}) = \mathcal{R} = \left(R_1, R, \overline{R}(\mu_{1R_1}, \mu_{2R}), L_R\right)$, where

(f) $R_1 = \mathfrak{f}_1(P_1) = \mathfrak{f}_1\left(\mathfrak{f}_1^{-1}(Q_1)\right) = Q_1$

(g) $R = \mathfrak{f}(P) = \mathfrak{f}\left(\mathfrak{f}^{-1}(Q)\right) = \mathfrak{f}\left(\mathfrak{f}^{-1}(V)\right) = V = Q$

(h) $\mu_{1R_1}: R_1 \longrightarrow L_V$ is defined by

$$\mu_{1R_1}y = \begin{cases} \mu_{1Q_1}\mathfrak{f}_1 x & , \text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x = \varphi \\ \mu_{2V}\mathfrak{f}_1 x \curlyvee \left[\mu_{1V_1}\mathfrak{f}_1 x \curlywedge \left(\underset{\substack{y = \mathfrak{f}_1 x \\ x \in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right)\right] & , y \in V \\ \mu_{1V_1}\mathfrak{f}_1 x \curlywedge \left(\underset{\substack{y = \mathfrak{f}_1 x \\ x \in P_1}}{\curlyvee} \Theta\mu_{1P_1}x\right) & , y \notin V \end{cases}$$

(i) $\mu_{2R}: R \longrightarrow L_V$ is defined by

$$\mu_{2R}y = \begin{cases} \mu_{2Q}\mathfrak{f}_1 x & , \text{whenever } \Theta^{-1\mu_{2Q}}\mathfrak{f}x = \varphi \\ \mu_{2V}\mathfrak{f}_1 x \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\underset{\substack{y = \mathfrak{f}x \\ x \in P}}{\curlyvee} \Theta\mu_{2P}x\right)\right] \end{cases}$$

(i) $L_R = \left(\left[\mu_{1R_1}(R_1)\right]\right)$, where $\left[\mu_{1R_1}(R_1)\right]$ = is the complete ideal generated by $\mu_{1R_1}(R_1)$ and $\left(\left[\mu_{1R_1}(R_1)\right]\right)$ isthe complete subalgebra generated by $\left[\mu_{1R_1}(R_1)\right]$

$\mathcal{R}$ and $\mathcal{Q}$ are full-equal nededi.e. Sufficient to show that,

(j) $R_1 = Q_1$

(k) $R = Q$

(l) $L_R = L_Q$

(m) $\mu_{1R_1}y = \mu_{1Q_1}y$

(n) $\mu_{2R}y = \mu_{2Q}y$

(f) and (g)⟹(j) and (k)

(e) and (j)⟹ (l)

(m): For $x \in U$,

$$\mu_{1R_1}y = \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]$$

$$= \mu_{2V}f_1x \vee \left[\mu_{1V_1}f_1x \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \Theta\left(\mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}f_1x \\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}f_1x\right)\right]\right)\right)\right]$$

$$= \mu_{2V}f_1x \vee \left[\mu_{1V_1}f_1x \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}f_1x \\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}f_1x\right)\right]\right)\right)\right]$$

Case: 1a: $\Theta^{-1}\mu_{1Q_1}f_1x = \varphi$, $\mu_{1R_1}f_1x = \mu_{1Q_1}f_1x$

Case: 1b: $\Theta^{-1}\mu_{1Q_1}f_1x \neq \varphi$

$$\mu_{1R_1}f_1x = \mu_{2V}f_1x \vee \left[\mu_{1V_1}f_1x \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}f_1x \\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}f_1x\right)\right]\right)\right)\right]$$

$$= \mu_{2V}f_1x \vee \left[\mu_{1V_1}f_1x \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \mu_{1Q_1}y\right]\right)\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \mu_{1Q_1}y\right)\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \mu_{1Q_1}y\right)\right]$$

$= \mu_{2V}y \vee \left(\mu_{1V_1}y \wedge \mu_{1Q_1}y\right) = \mu_{2V}y \vee \mu_{1Q_1}y = \mu_{1Q_1}y$,whenever $x \in U$

For $y \notin V$,

$$\mu_{1R_1}y = \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \Theta\mu_{1P_1}x\right)$$

$$= \mu_{1V_1}f_1x \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \Theta\left(\mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}f_1x \\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}f_1x\right)\right]\right)\right)$$

$$= \mu_{1V_1}f_1x \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in P_1}} \left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}f_1x \\ \alpha\in L_U}} \Theta\Theta^{-1}\mu_{1Q_1}f_1x\right)\right]\right)\right)$$

Case: 2a: $\Theta^{-1}\mu_{1Q_1}f_1x = \varphi$,$\mu_{1R_1}f_1x = \mu_{1Q_1}f_1x$

Case: 2b: $\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x \neq \varphi$

$$\mu_{1R_1}y = \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}\left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x\\ \alpha\in L_U}}\Theta\Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1 x\right)\right]\right)\right)$$

$$= \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}\left(\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \mu_{1Q_1}y\right]\right)\right)$$

$$= \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}\left(\Theta\mu_{2U}x \vee \mu_{1Q_1}\mathfrak{f}_1 x\right)\right)$$

$$= \mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x\\ x\in P_1}}\mu_{1Q_1}\mathfrak{f}_1 x\right) = \mu_{1V_1}y \wedge \mu_{1Q_1}y = \mu_{1Q_1}y$$

$$\mu_{2R}y = \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x\\ x\in P}}\Theta\mu_{2P}x\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x\\ x\in P}}\Theta\left\{\mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x\\ \alpha\in L_U}}\Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right]\right\}\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x\\ x\in P}}\left\{\Theta\mu_{2U}x \vee \left[\Theta\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x\\ \alpha\in L_U}}\Theta\Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right]\right\}\right)\right]$$

Case: 3a: $\Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi, \mu_{2R}\mathfrak{f}x = \mu_{2Q}\mathfrak{f}x$

Case: 3b: $\Theta^{-1}\mu_{2Q}\mathfrak{f}x \neq \varphi$

$$\mu_{2R}y = \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x\\ x\in P}}\left\{\{\Theta\mu_{2U}x \vee [\Theta\mu_{1U_1}x \wedge \mu_{2Q}\mathfrak{f}x]\}\right\}\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x\\ x\in P}}\{\Theta\mu_{2U}x \vee \mu_{2Q}y\}\right)\right]$$

$$= \mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x\\ x\in P}}\mu_{2Q}\mathfrak{f}x\right)\right] = \mu_{2V}y \vee [\mu_{1V_1}y \wedge \mu_{2Q}y] = \mu_{2V}y \vee \mu_{2Q}y = \mu_{2Q}y$$

Hence $\mathcal{R} = \mathcal{Q}$ i.e. the full-equality of $\bar{\mathfrak{f}}\bar{\mathfrak{f}}^{-1}(\mathcal{Q})$ and $\mathcal{Q}$ follows

2.2.2 Proposition: Suppose $\bar{\mathfrak{f}}_i: \mathcal{U} \longrightarrow \mathcal{V}$ with $\mathcal{P} \sqsubseteq \mathcal{U}$, P = U. Then

$$\bar{\mathfrak{f}}^{-1}\bar{\mathfrak{f}}(\mathcal{P}) \sqsupseteq \mathcal{P}$$

Proof: Let $\mathcal{P} \sqsubseteq \mathcal{U}$ and $\bar{\mathfrak{f}}: \mathcal{U} \longrightarrow \mathcal{V}$ be an Fs-set function, where

$$\mathcal{U} = (U_1, U, \bar{U}(\mu_{1U_1}, \mu_{2U}), L_U), \mathcal{V} = (V_1, V, \bar{V}(\mu_{1V_1}, \mu_{2V}), L_V), \mathcal{P} = (P_1, P, \bar{P}(\mu_{1P_1}, \mu_{2P}), L_P),$$

$P = U$ and $\mathfrak{f} = \mathfrak{f}_1|_U^V: U \longrightarrow V$ be onto.

Define $\bar{\mathfrak{f}}(\mathcal{P})$ as follows

$\bar{\mathfrak{f}}(\mathcal{P}) = \mathcal{Q} = \left(Q_1, Q, \bar{Q}(\mu_{1Q_1}, \mu_{2Q})L_Q\right)$, Where

(1) $Q_1 = \mathfrak{f}_1(P_1)$

(2) $Q = \mathfrak{f}(P)$

(3) $\mu_{1Q_1}: Q_1 \longrightarrow L_V$ is defined by

$$\mu_{1Q_1}y = \begin{cases} \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x \\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right], \text{if } y \in V \\ \mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x \\ x\in P_1}} \Theta\mu_{1P_1}x\right), \text{if } y \notin V \end{cases}$$

(4) $\mu_{2Q}: Q \longrightarrow L_V$ is defined by

$$\mu_{2Q}y = \mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\mu_{2P}x\right)\right]$$

(5) $L_Q = \left(\left[\mu_{1Q_1}(Q_1)\right]\right)$, where $\left[\mu_{1Q_1}(Q_1)\right]$is the complete ideal generated by $\mu_{1Q_1}(Q_1)$ and $\left(\left[\mu_{1Q_1}(Q_1)\right]\right)$ is the complete subalgebra generated by $\left[\mu_{1Q_1}(Q_1)\right]$

Again suppose $\bar{\mathfrak{f}}^{-1}\bar{\mathfrak{f}}(\mathcal{P}) = \bar{\mathfrak{f}}^{-1}(\mathcal{Q}) = \mathcal{R} = \left(R_1, R, \bar{R}(\mu_{1R_1,}\mu_{2R}), L_R\right)$, where

(a) $R_1 = \mathfrak{f}_1^{-1}(Q_1) = \mathfrak{f}_1^{-1}\left(\mathfrak{f}_1(P_1)\right) \sqsupseteq P_1$

(b) $R = \mathfrak{f}^{-1}(Q) = \mathfrak{f}^{-1}\left(\mathfrak{f}(P)\right) = \mathfrak{f}^{-1}\left(\mathfrak{f}(U)\right) = U = P$

(c) $\mu_{1R_1}: R_1 \longrightarrow L_R$ is defined by

$$\mu_{1R_1}x = \begin{cases} \mu_{1U_1}x & , \text{whenever } \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x \\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right)\right] & , x \in U \\ \mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x \\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\mathfrak{f}_1x\right) & , x \notin U \end{cases}$$

(d) $\mu_{2R}: R \longrightarrow L_R$ is defined by

$$\mu_{2R}x = \begin{cases} \mu_{2U}x & , \text{whenever } \Theta^{-1}\mu_{2Q}\mathfrak{f}x = \varphi \\ \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}x\right)\right] \end{cases}$$

(e) $L_R = L_U$

Now$\mathcal{R} \sqsupseteq \mathcal{P}$is needed i.e. to show

(f) $R_1 \sqsupseteq P_1$, $R \sqsubseteq P$

(g) $L_R \geq L_P$

(h) $\mu_{1R_1}|P_1 \geq \mu_{1P_1}, \mu_{2R} \leq \mu_{2P}|R$

(a) and (b) $\Longrightarrow$(f)

(e) $\Longrightarrow$ (g)

Need to show (h): For $x \in U$

$$\mu_{1R_1}x = \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\,\mathfrak{f}_1x\right)\right]$$

$$= \mu_{2U}x \curlyvee \left[\mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]\right)\right)\right]$$

$$\geq \mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]\right)\right)$$

$$\geq \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]\right)\right)$$

$$= \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right)\right)$$

$$= \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right)$$

$$= \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\Theta\left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \mu_{1P_1}x\right)\right) \geq \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \mu_{1P_1}x\right) = \mu_{1P_1}x$$

So that, $\mu_{1U_1}x \geq \mu_{1P_1}x$

For $x \notin U$

$$\mu_{1R_1}x = \mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\mu_{1Q_1}\,\mathfrak{f}_1x\right)$$

$$= \mu_{1U_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]\right)\right)$$

$$\geq \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left[\mu_{1V_1}y \curlywedge \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right]\right)\right)$$

$$= \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \curlyvee \left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right)\right)$$

$$= \mu_{1P_1}x \curlywedge \left(\curlyvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1x\\ \alpha\in L_U}} \Theta^{-1}\left(\curlyvee_{\substack{y=\mathfrak{f}_1x\\ x\in P_1}} \Theta\mu_{1P_1}x\right)\right)$$

$$= \mu_{1P_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{1Q_1}\mathfrak{f}_1 x \\ \alpha\in L_U}} \Theta^{-1}\Theta\left(\bigvee_{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}} \mu_{1P_1}x \right)\right)$$

$$\geq \mu_{1P_1}x \wedge \left(\bigvee_{\substack{y=\mathfrak{f}_1 x \\ x\in P_1}} \mu_{1P_1}x \right) = \mu_{1P_1}x,$$

So that, $\mu_{1R_1}x \geq \mu_{1P_1}x$

$$\mu_{2R}x = \mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\mu_{2Q}\mathfrak{f}x \right)\right]$$

$$= \mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{2V}y \vee \left[\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\mu_{2P}x \right)\right]\right)\right)\right]$$

$$= \mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\left(\mu_{1V_1}y \wedge \left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\mu_{2P}x \right)\right)\right)\right]$$

$$= \mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \Theta\mu_{2P}x \right)\right)\right]$$

$$= \mu_{2U}x \vee \left[\mu_{1U_1}x \wedge \left(\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\Theta\left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \mu_{2P}x \right)\right)\right]$$

$$= \mu_{2U}x \vee \left[\bigvee_{\substack{\Theta\alpha=\mu_{2Q}\mathfrak{f}x \\ \alpha\in L_U}} \Theta^{-1}\Theta\left(\bigvee_{\substack{y=\mathfrak{f}x \\ x\in P}} \mu_{2P}x \right)\right]$$

$$= \mu_{2U}x \vee \mu_{2P}x = \mu_{2P}x$$

Hence $\bar{\mathfrak{f}}^{-1}\bar{\mathfrak{f}}(\mathcal{P}) \sqsupseteq \mathcal{P}$.