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On Prime Spectrum of an ADL

R. Vasu Babu[1], Ch.Prabakara Rao [2] & Phani Yedlapalli [3]
[1&3]Department of Mathematics, Shri Vishnu Engineering College for Women, Vishnupur,Bhimavaram,
[2]Dadi Institue of Engineering and Technology, Anakapalle-
INDIA
Email: vasukgr@gmail.com ,
Email: raoprabhakar143@gmail.com
Email: phaniyedlapalli23@gmail.com

Abstract: In this paper, we equip the set of all prime ideals of an ADL with a suitable topology and call the resulting topological space the prime spectrum. We discuss certain important properties of the prime spectrum of a general ADL and obtain a necessary and sufficient condition, in terms of prime ideals and minimal prime ideals, for a pseudo-complemented ADL to be a Stone ADL. The discussion on the prime spectrum in this paper facilitates the proof of a characterization theorem for Stone ADL's We study the space of prime ideals of an *ADLA* together withthe Hull-Kernel topology, which is denoted by ***specA***. It is proved here that ***specA*** has an open base consisting of compact open sets and that ***specA*** is compact if and only if *A* has a maximal element.Also, we have obtained certain necessary and sufficient conditions, interms of topological properties of ***specA***, for an *ADL* to be a pseudo-complemented *ADL* or to be a Stone *ADL*.

AMS Subject Classification: 06D99

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Introduction

It is well known that a complemented distributive lattice is called a Booleanalgebra and a ring with unity, in which every element is an idempotent, iscalled a Boolean ring. M. H. Stone [3] has proved that any Boolean algebra can be made into a Boolean ring and vice-versa. U. M. Swamy and G. C. Rao[4] have introduced the notion of an Almost Distributive

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Lattice(abbreviatedas*ADL*) is a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras (Boolean rings). In this paper, we equip the set of all prime ideals of an *ADL* with a suitable topology and call the resulting topological space the prime spectrum. We discuss certain important properties of the prime spectrum of a general *ADL* and obtain a necessary and sufficient condition, in terms of prime ideals and minimal prime ideals, for a pseudo-complemented *ADL* to be a Stone *ADL*.

2 Preliminaries

In this section, we recall certain definitions and important results from [4, 6, 7, 8], those will be required in this paper.

Definition 2.1. A system $(A, \wedge, \vee, 0)$ is called an Almost Distributed Lattice (*ADL*) if A is a non-empty set, 0 is a distinguished element in A and $\wedge$, $\vee$ are binary operations on A satisfying the following axioms for any a , b and c in A .

(1) $0 \wedge a = 0$

(2) $a \vee 0 = a$

(3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(6) $(a \vee b) \wedge b = a$

Each of the axioms (1) through (6) above is independent from others. The element 0 is called the zero element. In the following we provide certain examples of *ADL*.

Example 2.2. Every distributive lattice bounded below is an *ADL* in whichthe least element is the zero element.

Several ring theoretic generalizations of Boolean algebras (other thanBoolean rings, whichw are precisely Boolean algebras) can be made as *ADL*'s. One such is the following.

Example 2.3. Let R be a commutative regular ring with unity (that is, R is a commutative ring with unity in which, for each $a \in R$, there exists an(unique) idempotent $a_0 \in R$ such that $aR = a_0R$. For any a and b in R, define $a \wedge b = a_0 b$ and $a \vee b = a + b + a_0 b$. Then $(R, \wedge, \vee, 0)$ is an ADL, where 0 is the zero element in the ring R.

Definition 2.4. For any Boolean algebra B, the Boolean space X_B given above is called the Stone space of B or the spectrum of B.

Theorem 2.5. *For any subset X of an ADL A,*

$$\langle X] = \left\{ \left(\bigvee_{i=1}^{n} x_i \right) \wedge a : n \geq 0, x_i \in X \text{ and } a \in A \right\}$$

Corollary2.6. *Let I be an ideal of an ADL A and $a \in A$ such that $a \notin I$. Then there exists a prime ideal P of A such that $I \subseteq P$ and $a \notin P$.*

Theorem2.7. *The following are equivalent to each other for any prime ideal P of an ADL A*

(1) *P is a minimal prime ideal of A*
(2) *$A - P$ is a maximal filter of A*
(3) *For any $a \in P, a^* \not\subset P$*

Theorem2.8. *The following are equivalent to each other for any ADL A.*

(1) *A is weakly pseudo-complemented.*
(2) *$\tilde{A}$ is a pseudo-complemented lattice.*

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(3) *A is pseudo-completed.*

3 Prime Spectrum of an Almost Distributive Lattice

In this section we discuss certain important properties of the prime spectrumof a general *ADL* and obtain a necessary and sufficient condition, in terms of prime ideals and minimal prime ideals, for a pseudo-complemented *ADL* to be a Stone *ADL*.

Definition3.1. Let $A=(A,\wedge,\vee,0)$ be a non-trivial Almost Distributive Lattice (*ADL*) and X be the set of all prime ideals of A. For any $a\in A$, define $X_a=\{P\in X : a\notin P\}$.

First note that, since A is non-trivial, X is a non-empty set. The proof of the following is a straight forward verification. Here after, throughout the remaining part of this paper, by an *ADL* we mean a nontrivial *ADL* only; for, then only prime ideals exist.

Theorem3.2. *Let X be the set of prime ideals of an ADL A. Then the following holds for any a and $b\in A$.*

(1) $X_a=\phi \Leftrightarrow a=0$

(2) $X_a=X \Leftrightarrow a$ *is maximal*

(3) $X_a\bigcap X_b=X_{a\wedge b}$

(4) $X_a\bigcup X_b=X_{a\vee b}$

(5) $X_a=X_b \Leftrightarrow a\,\square\, b \Leftrightarrow \langle a]=\langle b]$

(6) *For any* $S\subseteq A$, $\bigcup_{s\in S}X_s=\{P\in X : S\not\subset P\}=\{P\in X:\langle S]\not\subset P\}$, *where* $\langle S]$ *is the ideal generated by* S *in* A.

By 3.2(3), it follows that the class of all $X_a^{'}s$ $a\in A$ forms a base for a topology on X.

Definition 3.3. Let X be the set of all prime ideals of an *ADL A* The topology on X for which $\{X_a : a\in A\}$ is a base is called

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the Stone topology or Hull-Kernel topology. The set X together with Stone topology is called the Stone space of A or prime spectrum of A or simply the spectrum of A and it is denoted by $spec\,A$.

Note that the elements in the topological space **spec** A are the prime ideals of A and the open sets in **spec** A are precisely of the form $X(I)=\bigcup_{a\in I} X_a=\{P\in specA: I\not\subset P\}$ for some ideal I of A. The name Hull-Kernel topology is justified by the following.

Theorem 3.4. *Let A be an ADL and $X=specA$. For any $Y\subseteq X$, the kernel of Y be defined by $K(Y)=\bigcap_{P\in Y} P$ and for any $B\subseteq A$ the hull of B be defined by $h(B)=\{P\in X: B\subseteq P\}$ then for any $Y\subseteq X$, the topological closure in spec A is precisely $h(K(Y))$, hull of the kernel of the Y.*

Proof: Let $Y\subseteq X$. First note that $h(B)=h(\langle B])=X-X(\langle B])$ and hence $h(B)$ is closed in X for all $B\subseteq A$. In particular, $h(K(Y))$ is closed in X and $Y\subseteq h(K(Y))$. On the other hand, suppose that F is a closed subset of X such that $Y\subseteq F$. Then $F-X(I)$ for some ideal I of A and $Y\cap X(I)=\phi$. Therefore $I\subseteq P$ for all $P\in Y$ which implies that $I\subseteq K(Y)$. Therefore $X(I)\cap h(K(Y))=\phi$ and hence $h(K(Y))\subseteq X-X(I)=F$. Thus $h(k(Y))$ is closure of Y in X.

Theorem3.5. *Let A be an ADL, $X=specA$ and $Y\subseteq X$. Then Y is a compact open subset of X if and only if $Y=X_a$ for some $a\in A$.*

Corollary3.6. *For any $a\in A$, X_a is a compact open subset of $specA$.*

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Corollary 3.7. *For any ADL A, $specA$ is compact if and only if A has a maximal element.*

Corollary 3.8. *Let A be a pseudo-complemented ADL. Then $specA$ is compact.*

Next we interpret certain properties of ADL's in term of topological properties of their prime spectra. First, we have the following.

Definition3.9. An $ADL\, A=(A,\wedge,\vee,0)$ is said to be relatively complemented if for any x, a in A with $x \leq a$, there exist $y \in A$ such that $x \wedge y = 0$ and $x \vee y = a$. (That is, the interval $[0,a]$ is a complemented lattice for all $a \in A$).

Theorem 3.10. *The following are equivalent to each other for any ADL A.*

(1) *A is relatively complemented.*

(2) *For any a and b in A with $a \leq b$, the interval $[a,b]$ is complemented.*

(3) *For any a and b in A, there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$.*

Proof. $(1) \Rightarrow (2)$:

Let a and $b \in A$ and $a \leq b$. Let $x \in [a,b]$; that is, $a \leq x \leq b$. By (1), there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y = b$. Since $x \wedge y = 0$, we have $y \wedge x = x \wedge y = 0$ and $y \vee x = x \vee y = b$. From this, it follows that $y \leq y \vee x = b$. Put $z = a \vee y$. Then $a \leq z \leq b$ and $x \wedge z = x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y) = a \vee 0 = a$ and

$$
\begin{aligned}
x \vee z &= x \vee a \vee y \\
&= x \vee y \vee a \quad (\text{since } y \vee a = a \vee y) \\
&= b \vee a \\
&= b
\end{aligned}
$$

Thus $[a,b]$ is a complemented lattice.

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$(2) \Rightarrow (3)$:

Let a and $b \in A$. Then $a \in [0, a \vee b]$. By (2), there exists $x \in [0, a \vee b]$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$.

$(3) \Rightarrow (1)$:

Let x and $a \in A$ with $x \leq a$. Then there exists, by (3), $y \in A$ such that $x \wedge y = 0$ and $x \vee y = x \vee a = a$. Thus A is relatively complemented.

First, let us recall that a topological space X is said to be a Housdorffspace if, for any $x \neq y \in X$, there exists open sets G and H and $G \cap H = \phi$. Also, a topological space X is said to be a T_1 -space if, for any $x \neq y \in X$, there exist open sets G and H in X such that $x \in G - H$ and $y \in H - G$. It is known that X is a T_1 -space if and only if any singleton subset of X is closed (equivalently, any finite subset of X is closed)

Theorem 3.11. *The following are equivalent to each other for any non-trivial ADL.*

(1) specA is a Housdorff space.

(2) specA is a T_1 -space.

(3) Every prime ideal of A is maximal.

(4) Every prime filter of A is maximal.

(5) A is relatively complemented.

Proof: $(1) \Rightarrow (2)$: is trivial

$(2) \Rightarrow (3)$:

Let P be a prime ideal of A . Then $P \in specA$. By (2) , $\{P\}$ is closed. By theorem 3.4,

$h(K(\{P\})) = \{\overline{P}\} = \{P\}$. But $K\{P\} = P$ only.Therefore $h(P) = \{P\}$. This means that P is the only prime ideal containing P .

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Since every proper ideal is contained in a prime ideal, it follows, that P must be a maximum ideal of A.

$(3)\Rightarrow(4)$:

Follows from the fact that, for any $\phi \neq P \subsetneq A$, P is a prime ideal of A, P is a prime ideal of A if and only if $A-P$ is a prime filter of A . Also each of (3) and (4) is equivalent to saying that every prime ideal (filter) is a minimal prime ideal (filter).

$(4)\Rightarrow(5)$:

Let x and $a \in A$ such that $x \leq a$. If $x=0$ (then we can take $y=a$) or $x=a$, we are through. Therefore, we can assume that $0<x<a$. Consider the annihilator ideal $(x)^{*}=\{y \in A : x \wedge y=0\}$. We know that $(x)^{*}$ is an ideal of A . Now, we prove that $a \in (x)^{*} \vee \langle x] \subseteq P$. Then, by **theorem 2.6,** there exists a prime ideal P of A such that $(x)^{*} \vee \langle x] \subseteq P$ and $a \notin P$. We have $x \in \langle x] \subseteq P$. Since P is a minimal prime ideal and $x \in P$, we should get, by **theorem 2.7,** that $(x)^{*} \not\subset P$. This is a contradiction, since $(x)^{*} \subseteq (x)^{*} \vee \langle x] \subseteq P$. Therefore, $a \in (x)^{*} \vee \langle x]$ and hence $a = y \vee (x \wedge z)$ for some $y \in (x)^{*}$ and $z \in A$.

Now, we have $y \leq y \vee (x \wedge z) = a$ and $x \wedge y = 0$ (since $y \in (x)^{*}$).

Also, $x \vee y = y \vee x$ (since $x \wedge y = 0 = y \wedge x$)

$= y \vee x \vee (x \wedge z)$

$= y \vee x \vee y \vee (x \wedge z)$ (since $y \vee x = y \vee x \vee y$)

$= y \vee x \vee a$

$= a$ (since $x \leq a$ *and* $y \leq a$)

Thus A is relatively complemented.

$(5)\Rightarrow(1)$:

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Suppose that A is relatively complemented. Let $P \neq Q \in specA$. First, we prove that $P \not\subset Q$ and $Q \not\subset P$. For suppose, if possible, that $P \subseteq Q$. Since $P \neq Q$, there exists $a \in Q$ such that $a \notin P$. Now,

$b \in A \Rightarrow$ there exist $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$

$$\Rightarrow x \in P \text{ (since } a \wedge x = 0 \in P \text{ and } a \notin P)$$
$$\Rightarrow a \vee x \in Q \text{ (since } a \in Q \text{ and } x \in P \subset Q)$$
$$\Rightarrow a \vee b \in Q$$
$$\Rightarrow b = (a \vee b) \wedge b \in Q$$

This implies that $Q = A$, which is a contradiction to the fact that any prime ideal must be a proper ideal. Thus $P \not\subset Q$ and similarly $Q \not\subset P$. This implies that any prime ideal is maximal. Therefore every prime ideal is a minimal prime ideal. Now, choose $x \in P$ such that $x \notin Q$.

Then, by theorem2.7, $(x)^* \not\subset P$, there exists $y \in (x)^*$ such that $y \notin P$. Now, $P \in X_y, Q \in X_x$ and $X_y \wedge X_x = X_{x \wedge y} = X_0 = \phi$ and X_x and X_y are open sets in $X = specA$. Thus $specA$ is a Hausdorffspace.

In the next theorem, we arrive at a necessary and sufficient condition in terms of topological properties of $specA$, for an *ADLA* to be pseudo-complemented.

Theorem 3.12.*Let A be an ADL and $X = specA$. Then A is pseudo-complemented if and only if the interior of the complement of X_a is compact in X for all $a \in A$ and, in this case, $X_{a^*} = (X - X_a)^\circ$ for all $a \in A$.*

Proof.Suppose that A is pseudo-complemented and $*$ is a pseudo-complementation on A.

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Let $a \in A$. Since $a^* \wedge a = 0$, we get that $X_{a^*} \cap X_a = X_{a^* \wedge a} = X_0 = \phi$ and there fore $X_{a^*} \subseteq X - X_a$, since X_{a^*} is open, we have $X_{a^*} \subseteq (X - X_a)^\circ$, the interior of $X - X_{a^*}$.
Also

$P \in (X - X_a)^\circ \Rightarrow \exists\ b \in A$ such that $P \in X_b \subseteq (X - X_a)^\circ$

$\Rightarrow P \in X_b \ and\ X_b \cap X_a = \phi$

$\Rightarrow P \in X_b \ and\ b \wedge a = 0 = a \wedge b$

$\Rightarrow b \notin P \ and\ a^* \wedge b = b$

$\Rightarrow a^* \notin P$

$\Rightarrow P \in X_{a^*}$

Therefore $(X - X_a)^\circ = X_{a^*}$ which is compact(by theorem3.5) . Conversely suppose that $(X - X_a)^\circ$ is compact in X for all $a \in A$. Again by theorem3.5, $(X - X_a)^\circ = X_{a^*}$ for some $a^* \in A$.

That is, for each $a \in A$, we can choose $a^* \in A$ such that $(X - X_a)^\circ = X_{a^*}$ Note that a^* need not be unique. We can use the axiom of choice for choosing are a^* for each $a \in A$. Now, $X_{a^*} \subseteq X - X_a$ and hence $\phi = X_{a^*} \cap X_a = X_{a^* \wedge a}$.
Therefore $a^* \wedge a = 0$. Also, for any $b \in A$,

$a \wedge b = 0 \ \Rightarrow \ X_b \cap X_a = \phi$

$\Rightarrow \ X_b \subseteq X - X_a$

$\Rightarrow \ X_b \subseteq (X - X_a)^\circ \quad (\because X_b \text{ is open})$

$\Rightarrow \ X_b \subseteq X_{a^*}$

$\Rightarrow \ X_{a^* \wedge b} = X_{a^*} \cap X_b = X_b$

$\Rightarrow a^* \wedge b \sim b \quad (by\ 3.2(5))$

$\Rightarrow a^* \wedge b = (a^* \wedge b) \wedge b = b$.

Therefore $*$ is a weak pseudo-complementation on A. By theorem2.8[3] A is pseudo-complemented.

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