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Preface

This book covers all areas of mathematics and computer science. The contributions by the authors include Hilbert-type integral inequality; weight function; equivalent statement; beta function; cloud computing; load balancing; optimal solution; artificial intelligence and machine learning techniques; instance-based learning; reinforcement learning; Datanode; Hadoop; weak cluster; equilibrium point; trajectories; Normal distribution; logistic distribution; exponential distribution; best linear unbiased estimation; Riccati equation; duffing equation; integro-differential equations; chaotic solution; differential transforms method; Runge-Kutta 4 (RK4) method; modified equations of Emden type; differential transforms method; Runge-Kutta 4 (RK4) method; Fs-Set; Fs-Subset; (Fs-Point; FsB-topological space and FsB-Hausdorff space; random variable; continuous probability distribution; artificial neural network; intelligent transport system; departure rate; density function; mean of the distribution; normalizing constant etc. This book contains various materials suitable for students, researchers and academicians in the field of mathematics and computer Science.

Equivalent Property of a Hilbert-Type Integral Inequality Related to the Beta Function in the Whole Plane

Bicheng Yang^{1*}, Dongmei Xin¹ and Aizhen Wang¹

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Abstract

By means of the technique of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane are obtained. The constant factor related the beta function is proved to be the best possible. As applications, the case of the homogeneous kernel, the operator expressions and a few corollaries are considered.

Keywords: Hilbert-type integral inequality; weight function; equivalent statement; operator; beta function.

2010 Mathematics Subject Classification: 26D15.

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$. We have the following well known Hardy-Hilbert's integral inequality (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.1)$$

were the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. For $p = q = 2$, (1.1) reduces to the well known Hilbert's integral inequality. By using the weight functions, some extensions of (1.1) were given by [2], [3]. A few Hilbert-type inequalities with the homogenous and nonhomogenous kernels were provided by [4]-[7]. In 2017, Hong [8] also gave two equivalent statements between a Hilbert-type inequalities with the general homogenous kernel and parameters. Some other kinds of Hilbert-type inequalities were obtained by [9]-[16].

In 2007, Yang [17] gave a Hilbert-type integral inequality in the whole plane as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda} dx dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x) dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y) dy \right)^{\frac{1}{2}}, \quad (1.2)$$

with the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ ($\lambda > 0$, $B(u, v)$ is the beta function) (see [18]). He et al. [19]-[23] proved a few Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

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In this paper, by means of the technique of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane similar to (1.2) are obtained. The constant factor related to the beta function is proved to be the best possible. As applications, the case of the homogeneous kernel, the operator expressions and a few corollaries are considered.

2 An Example and Two Lemmas

Example 2.1. For $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = (0, \infty)$, we set $h(u) := \frac{(\max\{u, 1\})^{\alpha+\beta}}{|u-1|^{\lambda+\alpha}(\min\{u, 1\})^\beta}$ ($u \in \mathbf{R}_+$), and then for $a, b \neq 0$,

$$h(e^{ax+by}) = \frac{(\max\{e^{ax+by}, 1\})^{\alpha+\beta}}{|e^{ax+by} - 1|^{\lambda+\alpha}(\min\{e^{ax+by}, 1\})^\beta} \quad (x, y \in \mathbf{R}). \quad (2.1)$$

For $\sigma, \mu > \beta, \sigma + \mu = \lambda < 1 - \alpha$ ($\alpha + 2\beta < 1$), in view of $h(v^{-1})v^{1-\sigma} = h(v)v^{\mu-1}$ ($0 < v < 1$), we find

$$\begin{aligned} k_\lambda(\sigma) &:= \int_0^\infty h(u)u^{\sigma-1}du = \int_0^1 h(u)(u^{\sigma-1} + u^{\mu-1})du \\ &= \int_0^1 \frac{(\max\{u, 1\})^{\alpha+\beta}}{(1-u)^{\lambda+\alpha}(\min\{u, 1\})^\beta} (u^{\sigma-1} + u^{\mu-1})du \\ &= \int_0^1 \frac{1}{(1-u)^{\lambda+\alpha}} (u^{\sigma-\beta-1} + u^{\mu-\beta-1})du \\ &= B(1-\lambda-\alpha, \sigma-\beta) + B(1-\lambda-\alpha, \mu-\beta) \in \mathbf{R}_+, \end{aligned} \quad (2.2)$$

where, $B(u, v) := \int_0^1 (1-t)^{u-1}t^{v-1}dt$ ($u, v > 0$) is the beta function (cf. [18]).

In particular, (i) for $\alpha = 0$, we have $\sigma, \mu > \beta, \sigma + \mu = \lambda < 1$ ($\beta < \frac{1}{2}$), $h_1(u) = \frac{(\max\{u, 1\})^\beta}{|u-1|^\lambda(\min\{u, 1\})^\beta}$ ($u > 0$),

and

$$k_\lambda^{(1)}(\sigma) = B(1-\lambda, \sigma-\beta) + B(1-\lambda, \mu-\beta);$$

(ii) for $\beta = 0$, we have $\sigma, \mu > 0, \sigma + \mu = \lambda < 1 - \alpha$ ($\alpha < 1$), $h_2(u) = \frac{(\max\{u, 1\})^\alpha}{|u-1|^{\lambda+\alpha}}$ ($u > 0$),

and

$$k_\lambda^{(2)}(\sigma) = B(1-\lambda-\alpha, \sigma) + B(1-\lambda-\alpha, \mu);$$

(iii) for $\beta = -\alpha$, we have $\sigma, \mu > -\alpha, \sigma + \mu = \lambda < 1 - \alpha$ ($\alpha > -1$), $h_3(u) = \frac{(\min\{u, 1\})^\alpha}{|u-1|^{\lambda+\alpha}}$ ($u > 0$),

and

$$k_\lambda^{(3)}(\sigma) = B(1-\lambda-\alpha, \sigma+\alpha) + B(1-\lambda-\alpha, \mu+\alpha).$$

In the case of (iii), for $\alpha = 0$, we have $\sigma, \mu > 0, \sigma + \mu = \lambda < 1$, $h_4(u) = \frac{1}{|u-1|^\lambda}$ ($u > 0$),

and

$$k_\lambda^{(4)}(\sigma) = B(1-\lambda, \sigma) + B(1-\lambda, \mu).$$

In the following, we assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a, b \neq 0, \sigma_1, \sigma \in \mathbf{R}, \sigma, \mu > \beta, \sigma + \mu = \lambda < 1 - \alpha$ ($\alpha + 2\beta < 1$), and

$$\begin{aligned} K_\lambda(\sigma) &: = \frac{1}{|a|^{1/q}|b|^{1/p}} k_\lambda(\sigma) = \frac{1}{|a|^{1/q}|b|^{1/p}} \\ &\quad \times (B(1-\lambda-\alpha, \sigma-\beta) + B(1-\lambda-\alpha, \mu-\beta)). \end{aligned} \quad (2.3)$$

For $n \in \mathbf{N} = \{1, 2, \dots\}$, we define two sets $E_c := \{t \in \mathbf{R}; ct \geq 0\}$, $F_c := \mathbf{R} \setminus E_c = \{t \in \mathbf{R}; ct < 0\}$ ($c = a, b$), and the following two expressions:

$$I_1 := \int_{F_b} e^{(\sigma_1 + \frac{1}{qn})by} \left[\int_{E_a} h(e^{ax+by}) e^{(\sigma - \frac{1}{pn})ax} dx \right] dy, \quad (2.4)$$

$$I_2 := \int_{E_b} e^{(\sigma_1 - \frac{1}{qn})by} \left[\int_{F_a} h(e^{ax+by}) e^{(\sigma + \frac{1}{pn})ax} dx \right] dy. \quad (2.5)$$

Setting $u = e^{ax+by}$ in (2.4), in view of Fubini theorem (cf. [24]), it follows that

$$\begin{aligned} I_1 &= \frac{1}{|a|} \int_{F_b} e^{(\sigma_1 - \sigma + \frac{1}{n})by} \left(\int_{e^{by}}^{\infty} (u) u^{\sigma - \frac{1}{pn} - 1} du \right) dy \\ &= \frac{1}{|ab|} \int_0^1 v^{\sigma_1 - \sigma + \frac{1}{n} - 1} \left(\int_v^{\infty} h(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dv \quad (v = e^{by}). \end{aligned} \quad (2.6)$$

In the same way, we find that

$$\begin{aligned} I_2 &= \frac{1}{|a|} \int_{E_b} e^{(\sigma_1 - \sigma - \frac{1}{n})by} \left(\int_0^{e^y} h(u) u^{\sigma + \frac{1}{pn} - 1} dx \right) dy \\ &= \frac{1}{|ab|} \int_1^{\infty} v^{\sigma_1 - \sigma - \frac{1}{n} - 1} \left(\int_0^v h(u) u^{\sigma + \frac{1}{pn} - 1} du \right) dv. \end{aligned} \quad (2.7)$$

Lemma 2.1. *If there exists a constant M , such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in \mathbf{R} , the following inequality*

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(e^{ax+by}) f(x) g(y) dx dy \\ &\leq M \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma_1 by}} \right)^q dy \right]^{\frac{1}{q}} \end{aligned} \quad (2.8)$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. (i) If $\sigma_1 < \sigma$, then for $n > \frac{1}{\sigma - \sigma_1}$ ($n \in \mathbf{N}$), we set two functions

$$f_n(x) := \begin{cases} e^{(\sigma - \frac{1}{pn})ax}, & x \in E_a \\ 0, & x \in F_a \end{cases}, \quad g_n(y) := \begin{cases} 0, & y \in E_b \\ e^{(\sigma_1 + \frac{1}{qn})by}, & y \in F_b \end{cases},$$

and obtain

$$\begin{aligned} J_2 &:= \left[\int_{-\infty}^{\infty} e^{-p\sigma ax} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} e^{-q\sigma_1 by} g_n^q(y) dy \right]^{\frac{1}{q}} \\ &= \left(\int_{E_a} e^{-\frac{ax}{n}} dx \right)^{\frac{1}{p}} \left(\int_{F_b} e^{\frac{by}{n}} dy \right)^{\frac{1}{q}} = \frac{n}{|a|^{1/p} |b|^{1/q}}. \end{aligned}$$

By (2.6) and (2.8), we find

$$\begin{aligned} &\frac{1}{|ab|} \int_0^1 v^{\sigma_1 - \sigma + \frac{1}{n} - 1} dv \int_1^{\infty} \frac{u^{\alpha + \beta + \sigma - \frac{1}{pn} - 1}}{(u - 1)^{\lambda + \alpha}} du \\ &\leq I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(e^{ax+by}) f_n(x) g_n(y) dx dy \leq M J_2 = \frac{Mn}{|a|^{1/p} |b|^{1/q}}. \end{aligned} \quad (2.9)$$

For any $n > \frac{1}{\sigma - \sigma_1}$ ($n \in \mathbf{N}$), $\sigma_1 - \sigma + \frac{1}{n} < 0$, it follows that $\int_0^1 v^{\sigma_1 - \sigma + \frac{1}{n} - 1} dv = \infty$. In view of $\int_1^\infty \frac{u^{\alpha + \beta + \sigma - \frac{1}{pn} - 1}}{(u-1)^{\lambda + \alpha}} du > 0$, by (2.9), we find that $\infty \leq \frac{Mn}{|a|^{1/p}|b|^{1/q}} < \infty$, which is a contradiction.

(ii) If $\sigma_1 > \sigma$, then for $n > \frac{1}{\sigma_1 - \sigma}$ ($n \in \mathbf{N}$), we set functions

$$\tilde{f}_n(x) := \begin{cases} 0, & x \in E_a \\ e^{(\sigma + \frac{1}{pn})ax}, & x \in F_a \end{cases}, \quad \tilde{g}_n(y) := \begin{cases} e^{(\sigma_1 - \frac{1}{qn})by}, & y \in E_b \\ 0, & y \in F_b \end{cases},$$

and find

$$\begin{aligned} \tilde{J}_2 & : = \left[\int_{-\infty}^{\infty} e^{-p\sigma ax} \tilde{f}_n(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} e^{-q\sigma_1 by} \tilde{g}_n(y) dy \right]^{\frac{1}{q}} \\ & = \left(\int_{F_a} e^{\frac{ax}{n}} dx \right)^{\frac{1}{p}} \left(\int_{E_b} e^{-\frac{by}{n}} dy \right)^{\frac{1}{q}} = \frac{n}{|a|^{1/p}|b|^{1/q}}. \end{aligned}$$

By (2.7) and (2.8), we obtain

$$\begin{aligned} & \frac{1}{|ab|} \int_1^\infty v^{\sigma_1 - \sigma - \frac{1}{n} - 1} dv \int_0^1 \frac{u^{\sigma - \beta + \frac{1}{pn} - 1}}{(1-u)^{\lambda + \alpha}} du \\ & \leq I_1 = \int_0^\infty \int_0^\infty h(e^{ax+by}) \tilde{f}_n(x) \tilde{g}_n(y) dx dy \leq M \tilde{J}_2 = \frac{Mn}{|a|^{1/p}|b|^{1/q}}. \end{aligned} \quad (2.10)$$

For $n > \frac{1}{\sigma_1 - \sigma}$ ($n \in \mathbf{N}$), $\sigma_1 - \sigma - \frac{1}{n} > 0$, it follows that $\int_1^\infty v^{\sigma_1 - \sigma - \frac{1}{n} - 1} dv = \infty$. By (2.10), in view of $\int_0^1 \frac{u^{\sigma - \beta + \frac{1}{pn} - 1}}{(1-u)^{\lambda + \alpha}} du > 0$, we have $\infty \leq \frac{Mn}{|a|^{1/p}|b|^{1/q}} < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

The lemma is proved. □

For $\sigma_1 = \sigma$, we have

Lemma 2.2. *If there exists a constant M , such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in \mathbf{R} , the following inequality*

$$\begin{aligned} I & = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(e^{ax+by}) f(x) g(y) dx dy \\ & \leq M \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}} \right)^q dy \right]^{\frac{1}{q}} \end{aligned} \quad (2.11)$$

holds true, then we have $M \geq K_\lambda(\sigma) (> 0)$.

Proof. By (2.6), for $\sigma_1 = \sigma$, we obtain

$$\begin{aligned} I_1 & = \frac{1}{|ab|} \int_0^1 v^{\frac{1}{n} - 1} \left(\int_v^1 h(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dv \\ & \quad + \frac{1}{|ab|} \int_0^1 v^{\frac{1}{n} - 1} \left(\int_1^\infty h(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dv \\ & = \frac{1}{|ab|} \int_0^1 \left(\int_0^u v^{\frac{1}{n} - 1} dv \right) h(u) u^{\sigma - \frac{1}{pn} - 1} du \\ & \quad + \frac{n}{|ab|} \int_1^\infty h(u) u^{\sigma - \frac{1}{pn} - 1} du \\ & = \frac{n}{|ab|} \left(\int_0^1 h(u) u^{\sigma + \frac{1}{qn} - 1} du \int_1^\infty h(u) u^{\sigma - \frac{1}{pn} - 1} du \right). \end{aligned}$$

We use inequality $I_1 \leq M\tilde{J}_2$ (for $\sigma_1 = \sigma$) as follows

$$\begin{aligned} & \frac{|a|^{1/p}|b|^{1/q}}{n} I_1 \\ = & \frac{1}{|a|^{1/q}|b|^{1/p}} \left(\int_0^1 H(u)u^{\sigma+\frac{1}{qn}-1} du + \int_1^\infty H(u)u^{\sigma-\frac{1}{pn}-1} du \right) \leq M. \end{aligned} \quad (2.12)$$

By Fatou lemma (cf. [24]) and (2.12), it follows that

$$\begin{aligned} K_\lambda(\sigma) &= \frac{1}{|a|^{1/q}|b|^{1/p}} \\ &\times \left(\int_0^1 \lim_{n \rightarrow \infty} h(u)u^{\sigma+\frac{1}{qn}-1} du + \int_1^\infty \lim_{n \rightarrow \infty} h(u)u^{\sigma-\frac{1}{pn}-1} du \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{|a|^{1/p}|b|^{1/q}}{n} I_1 \leq M. \end{aligned}$$

The lemma is proved. □

3 Main Results and Some Corollaries

Theorem 3.1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) For any nonnegative measurable function $f(x)$ in \mathbf{R} , we have the following inequality:

$$\begin{aligned} J &: = \left[\int_{-\infty}^\infty e^{p\sigma_1 by} \left(\int_{-\infty}^\infty h(e^{ax+by}) f(x) dx \right)^p dy \right]^{\frac{1}{p}} \\ &\leq M \left[\int_{-\infty}^\infty \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}}. \end{aligned} \quad (3.1)$$

(ii) For any nonnegative measurable functions $f(x)$ and $g(y)$ in \mathbf{R} , we have the following inequality:

$$\begin{aligned} I &= \int_{-\infty}^\infty \int_{-\infty}^\infty h(e^{ax+by}) f(x) g(y) dx dy \\ &\leq M \left[\int_{-\infty}^\infty \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty \left(\frac{g(y)}{e^{\sigma_1 by}} \right)^q dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.2)$$

(iii) $\sigma_1 = \sigma$, and $M \geq K_\lambda(\sigma) (> 0)$.

Proof. (i) \Rightarrow (ii). By Hölder's inequality (see [25]), we have

$$\begin{aligned} I &= \int_{-\infty}^\infty \left(e^{\sigma_1 by} \int_{-\infty}^\infty h(e^{ax+by}) f(x) dx \right) \left(e^{-\sigma_1 by} g(y) \right) dy \\ &\leq J \left[\int_{-\infty}^\infty \left(\frac{g(y)}{e^{\sigma_1 by}} \right)^q dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.3)$$

Then by (3.1), we have (3.2).

(ii) \Rightarrow (iii). By Lemma 1, we have $\sigma_1 = \sigma$. Then by Lemma 2, we have $M \geq K_\lambda(\sigma) (> 0)$.

(iii) \Rightarrow (i). Setting $u = e^{ax+by}$, we obtain the following weight functions: For $y, x \in \mathbf{R}$,

$$\begin{aligned} \omega(\sigma, y) &: = e^{\sigma by} \int_{-\infty}^{\infty} h(e^{ax+by}) e^{\sigma ax} dx \\ &= \frac{1}{|a|} \int_0^{\infty} h(u) u^{\sigma-1} du = \frac{1}{|a|} k_{\lambda}(\sigma), \end{aligned} \quad (3.4)$$

$$\varpi(\sigma, x) : = e^{\sigma ax} \int_{-\infty}^{\infty} h(e^{ax+by}) e^{\sigma by} dy = \frac{1}{|b|} k_{\lambda}(\sigma). \quad (3.5)$$

By Hölder's inequality with weight and (3.4), we have

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} h(e^{ax+by}) f(x) dx \right)^p \\ &= \left[\int_{-\infty}^{\infty} h(e^{ax+by}) \left(\frac{e^{\sigma by/p}}{e^{\sigma ax/q}} f(x) \right) \left(\frac{e^{\sigma ax/q}}{e^{\sigma by/p}} \right) dx \right]^p \\ &\leq \int_{-\infty}^{\infty} h(e^{ax+by}) \frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x) dx \left(\int_{-\infty}^{\infty} h(e^{ax+by}) \frac{e^{\sigma ax}}{e^{\sigma byq/p}} dx \right)^{p/q} \\ &= \left[\omega(\sigma, y) e^{-q\sigma by} \right]^{p-1} \int_{-\infty}^{\infty} h(e^{ax+by}) \frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x) dx \\ &= \left(\frac{1}{|a|} k(\sigma) \right)^{p-1} e^{-p\sigma by} \int_{-\infty}^{\infty} h(e^{ax+by}) \frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x) dx. \end{aligned} \quad (3.6)$$

For $\sigma_1 = \sigma$, by Fubini theorem (see [24]) and (3.5), we have

$$\begin{aligned} J &\leq \left(\frac{1}{|a|} k_{\lambda}(\sigma) \right)^{\frac{1}{q}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(e^{ax+by}) \frac{e^{\sigma by}}{e^{\sigma axp/q}} f^p(x) dx dy \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{|a|} k_{\lambda}(\sigma) \right)^{\frac{1}{q}} \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(e^{ax+by}) \frac{e^{\sigma by}}{e^{\sigma axp/q}} dy \right) f^p(x) dx \right]^{\frac{1}{p}} \\ &= \left(\frac{1}{|a|} k_{\lambda}(\sigma) \right)^{\frac{1}{q}} \left(\int_{-\infty}^{\infty} \omega(\sigma, x) e^{-p\sigma ax} f^p(x) dx \right)^{\frac{1}{p}} \\ &= K_{\lambda}(\sigma) \left(\int_{-\infty}^{\infty} e^{-p\sigma ax} f^p(x) dx \right)^{\frac{1}{p}}. \end{aligned}$$

For $K_{\lambda}(\sigma) \leq M$, we have (3.1).

Therefore, the statements (i), (ii) and (iii) are equivalent.

The theorem is proved. □

Theorem 3.2. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx < \infty$, we have the following inequality:

$$\begin{aligned} J_1 &= \left\{ \int_{-\infty}^{\infty} e^{p\sigma by} \left[\int_{-\infty}^{\infty} \frac{(\max\{e^{ax+by}, 1\})^{\alpha+\beta} f(x) dx}{|e^{ax+by} - 1|^{\lambda+\alpha} (\min\{e^{ax+by}, 1\})^{\beta}} \right]^p dy \right\}^{\frac{1}{p}} \\ &< K_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}}. \end{aligned} \quad (3.7)$$

(ii) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$ and $g(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}}\right)^q dy < \infty$, we have the following inequality:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\max\{e^{ax+by}, 1\})^{\alpha+\beta} f(x)g(y)}{|e^{ax+by} - 1|^{\lambda+\alpha} (\min\{e^{ax+by}, 1\})^{\beta}} dx dy \\ &< K_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}}\right)^q dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.8)$$

Moreover, the constant factor $K_{\lambda}(\sigma)$ in (3.7) and (3.8) is the best possible.

In particular, for $\alpha = \beta = 0, \sigma, \mu > 0, \sigma + \mu = \lambda < 1$

$$\tilde{K}_{\lambda}(\sigma) := \frac{1}{|a|^{1/q}|b|^{1/p}} (B(1-\lambda, \sigma) + \zeta(1-\lambda, \mu)), \quad (3.9)$$

we have the following equivalent inequalities with the best possible constant factor $\tilde{K}_{\lambda}(\sigma)$:

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} e^{p\sigma by} \left(\int_{-\infty}^{\infty} \frac{f(x)}{|e^{ax+by} - 1|^{\lambda}} dx \right)^p dy \right]^{\frac{1}{p}} \\ &< \tilde{K}_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{|e^{ax+by} - 1|^{\lambda}} dx dy \\ &< \tilde{K}_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}}\right)^q dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.11)$$

Proof. We first prove that (3.7) is valid. If (3.6) takes the form of equality for a $y \in \mathbf{R}$, then (see [25]), there exists constants A and B , such that they are not all zero and

$$A \frac{e^{\sigma by}}{e^{\sigma ax p/q}} f^p(x) = B \frac{e^{\sigma ax}}{e^{\sigma by q/p}} \text{ a.e. in } \mathbf{R}.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$\left(\frac{f(x)}{e^{\sigma ax}}\right)^p = e^{-q\sigma by} \frac{B}{A} \text{ a.e. in } \mathbf{R},$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$. Hence, (3.6) takes the form of strict inequality. For $\sigma_1 = \sigma$ by the proof of Theorem 1, we obtain (3.7).

(i) \Rightarrow (ii). By (3.3) (for $\sigma_1 = \sigma$) and (3.7), we have (3.8).

(ii) \Rightarrow (i). We set the following function:

$$g(y) := e^{p\sigma by} \left(\int_{-\infty}^{\infty} h(e^{ax+by}) f(x) dx \right)^{p-1} \quad (y \in \mathbf{R}).$$

If $J_1 = \infty$, then it is impossible since (3.7) is valid; if $J_1 = 0$, then (3.7) is trivially valid. In the

following, we suppose that $0 < J_1 < \infty$. By (3.8), we have

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}} \right)^q dy = J_1^p = I \\ &< K_\lambda(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}} \right)^q dy \right]^{\frac{1}{q}} < \infty, \\ J_1 &= \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\sigma by}} \right)^q dy \right]^{\frac{1}{p}} < K_\lambda(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}}, \end{aligned}$$

namely, (3.7) follows, which is equivalent to (3.8).

Hence, Statements (i) and (ii) are valid and equivalent.

If there exists a constant $M \leq K_\lambda(\sigma)$, such that (3.8) is valid when replacing $K_\lambda(\sigma)$ by M , then by Lemma 2, we have $K_\lambda(\sigma) \leq M$. Hence, the constant factor $M = K_\lambda(\sigma)$ in (3.8) is the best possible.

The constant factor $K_\lambda(\sigma)$ in (3.7) is still the best possible. Otherwise, by (3.3) (for $\sigma_1 = \sigma$), we would reach a contradiction that the constant factor $K_\lambda(\sigma)$ in (3.8) is not the best possible.

The theorem is proved. □

For $g(y) = e^{-\lambda by} G(y)$, and $\mu_1 = \lambda - \sigma_1$ in Theorem 1 and Theorem 2, then replacing $b(G(y))$ by $-b(g(y))$, setting

$$k_\lambda(e^{ax}, e^{by}) := \frac{(\max\{e^{ax}, e^{by}\})^{\alpha+\beta}}{|e^{ax} - e^{by}|^{\lambda+\alpha} (\min\{e^{ax}, e^{by}\})^\beta} (x, y \in \mathbf{R}), \quad (3.12)$$

we have the following corollaries:

Corollary 3.1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) *For any nonnegative measurable function $f(x)$ in \mathbf{R} , we have the following inequality:*

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} e^{p\mu_1 by} \left(\int_{-\infty}^{\infty} k_\lambda(e^{ax}, e^{by}) f(x) dx \right)^p dy \right]^{\frac{1}{p}} \\ &\leq M \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}}. \end{aligned} \quad (3.13)$$

(ii) *For any nonnegative measurable functions $f(x)$ and $g(y)$ in \mathbf{R} , we have the following inequality:*

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_\lambda(e^{ax}, e^{by}) f(x) g(y) dx dy \\ &\leq M \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}} \right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\mu_1 by}} \right)^q dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.14)$$

(iii) $\mu_1 = \mu$, and $M \geq K_\lambda(\sigma) (> 0)$.

Corollary 3.2. *The following statements (i) and (ii) are valid and equivalent:*

(i) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$, we have the following inequality:

$$\left\{ \int_{-\infty}^{\infty} e^{pb\mu y} \left[\int_{-\infty}^{\infty} \frac{(\max\{e^{ax}, e^{by}\})^{\alpha+\beta} f(x)}{|e^{ax} - e^{by}|^{\lambda+\alpha} (\min\{e^{ax}, e^{by}\})^{\beta}} dx \right]^p dy \right\}^{\frac{1}{p}} < K_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}}. \quad (3.15)$$

(ii) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx < \infty$, and $g(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\mu by}}\right)^q dy < \infty$, we have the following inequality:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\max\{e^{ax}, e^{by}\})^{\alpha+\beta} f(x)g(y)}{|e^{ax} - e^{by}|^{\lambda+\alpha} (\min\{e^{ax}, e^{by}\})^{\beta}} dx dy < K_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\mu by}}\right)^q dy \right]^{\frac{1}{q}}. \quad (3.16)$$

Moreover, the constant factor $K_{\lambda}(\sigma)$ in (3.15) and (3.16) is the best possible.

In particular, for $\alpha = \beta = 0, \sigma, \mu > 0, \sigma + \mu = \lambda < 1$, we have the following equivalent inequalities with the best possible constant factor $\tilde{K}_{\lambda}(\sigma)$:

$$\left[\int_{-\infty}^{\infty} e^{p\mu by} \left(\int_{-\infty}^{\infty} \frac{f(x)}{|e^{ax} - e^{by}|^{\lambda}} dx \right)^p dy \right]^{\frac{1}{p}} < \tilde{K}_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}}, \quad (3.17)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{|e^{ax} - e^{by}|^{\lambda}} dx dy < \tilde{K}_{\lambda}(\sigma) \left[\int_{-\infty}^{\infty} \left(\frac{f(x)}{e^{\sigma ax}}\right)^p dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \left(\frac{g(y)}{e^{\mu by}}\right)^q dy \right]^{\frac{1}{q}}. \quad (3.18)$$

In (3.10) and (3.11), setting $F(x) = e^{\frac{\lambda a}{2}x} f(x), G(y) = e^{\frac{\lambda b}{2}y} g(y)$, then replacing back $F(x)$ ($G(y)$) by $f(x)$ ($g(y)$), and introducing the hyperbolic sine function as $\sinh(s) = \frac{e^s - e^{-s}}{2}$, we have

Corollary 3.3. If $\sigma, \mu > 0, \sigma + \mu = \lambda < 1$, then the following statements (i) and (ii) are valid and equivalent:

(i) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x) \right]^p dx < \infty$, we have the following inequality:

$$\left[\int_{-\infty}^{\infty} e^{p(\sigma-\frac{\lambda}{2})by} \left(\int_{-\infty}^{\infty} \frac{f(x)}{|\sinh(\frac{ax+by}{2})|^{\lambda}} dx \right)^p dy \right]^{\frac{1}{p}} < 2\tilde{K}_{\lambda}(\sigma) \left\{ \int_{-\infty}^{\infty} \left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x) \right]^p dx \right\}^{\frac{1}{p}}. \quad (3.19)$$

(ii) For any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} \left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x) \right]^p dx < \infty$ and $g(y) \geq 0$, satisfying

$0 < \int_{-\infty}^{\infty} \left[e^{(\frac{\lambda}{2}-\sigma)by} g(y) \right]^q dy < \infty$, we have the following inequality:

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{|\sinh(\frac{ax+by}{2})|^\lambda} dx dy \\ & < 2\tilde{K}_\lambda(\sigma) \left\{ \int_{-\infty}^{\infty} \left[e^{(\frac{\lambda}{2}-\sigma)ax} f(x) \right]^p dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{\infty} \left[e^{(\frac{\lambda}{2}-\sigma)by} g(y) \right]^q dy \right\}^{\frac{1}{q}}. \end{aligned} \quad (3.20)$$

Moreover, the constant factor $2\tilde{K}_\lambda(\sigma)$ in (3.19) and (3.20) is the best possible.

4 Operator Expressions

We set the following functions: $\varphi(x) := e^{-p\sigma ax}$, $\psi(y) := e^{-q\sigma by}$, $\phi(y) := e^{-q\mu by}$, wherefrom, $\psi^{1-p}(y) = e^{p\sigma by}$, $\phi^{1-p}(y) = e^{p\mu by}$ ($x, y \in \mathbf{R}$), and define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}) := \left\{ f : \|f\|_{p,\varphi} := \left(\int_{-\infty}^{\infty} \varphi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$\begin{aligned} L_{q,\psi}(\mathbf{R}) &= \left\{ g : \|g\|_{q,\psi} := \left(\int_{-\infty}^{\infty} \psi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{q,\phi}(\mathbf{R}) &= \left\{ g : \|g\|_{q,\phi} := \left(\int_{-\infty}^{\infty} \phi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{p,\psi^{1-p}}(\mathbf{R}) &= \left\{ h : \|h\|_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\phi^{1-p}}(\mathbf{R}) &= \left\{ h : \|h\|_{p,\phi^{1-p}} = \left(\int_{-\infty}^{\infty} \phi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$h_1(y) := \int_{-\infty}^{\infty} h(e^{ax+by}) f(x) dx \quad (y \in \mathbf{R}),$$

by (3.9), we have

$$\|h_1\|_{p,\psi^{1-p}} = \left[\int_{-\infty}^{\infty} \psi^{1-p}(y) H_1^p(y) dy \right]^{\frac{1}{p}} \leq K_\lambda(\sigma) \|f\|_{p,\varphi} < \infty. \quad (4.1)$$

Definition 4.1. Define a Hilbert-type integral operator with the nonhomogeneous kernel $T^{(1)} : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying for any $y \in \mathbf{R}$, $T^{(1)}f(y) = h_1(y)$.

In view of (4.1), it follows that

$$\|T^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq K_\lambda(\sigma) \|f\|_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R})} \frac{\|T^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq K_\lambda(\sigma).$$

If we define the formal inner product of $T^{(1)}f$ and g as follows:

$$(T^{(1)}f, g) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(e^{ax+by})f(x)dx \right) g(y)dy,$$

then we can rewrite Theorem 2 as follows:

Theorem 4.1. *The following statements (i) and (ii) are valid and equivalent:*

(i) *For any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, we have the following inequality:*

$$\|T^{(1)}f\|_{p,\psi^{1-p}} < K_{\lambda}(\sigma)\|f\|_{p,\varphi}. \quad (4.2)$$

(ii) *For any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\psi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, and $\|g\|_{q,\psi} > 0$, we have the following inequality:*

$$(T^{(1)}f, g) < K_{\lambda}(\sigma)\|f\|_{p,\varphi}\|g\|_{q,\psi}. \quad (4.3)$$

Moreover, the constant factor $K_{\lambda}(\sigma)$ in (4.2) and (4.3) is the best possible, namely,

$$\|T^{(1)}\| = K_{\lambda}(\sigma).$$

(b) In view of Corollary 2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$h_2(y) := \int_{-\infty}^{\infty} k_{\lambda}(e^{ax}, e^{by})f(x)dx \quad (y \in \mathbf{R}),$$

by (3.15), we have

$$\|h_2\|_{p,\phi^{1-p}} = \left[\int_{-\infty}^{\infty} \phi^{1-p}(y)h_2^p(y)dy \right]^{\frac{1}{p}} \leq K_{\lambda}(\sigma)\|f\|_{p,\varphi} < \infty. \quad (4.4)$$

Definition 4.2. *Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)} : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\phi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R})$, satisfying for any $y \in \mathbf{R}, T^{(2)}f(y) = h_2(y)$.*

In view of (4.4), it follows that

$$\|T^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq K_{\lambda}(\sigma)\|f\|_{p,\varphi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$\|T^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R})} \frac{\|T^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq K_{\lambda}(\sigma).$$

If we define the formal inner product of $T^{(2)}f$ and g as follows:

$$(T^{(2)}f, g) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} k_{\lambda}(e^{ax}, e^{by})f(x)dx \right) g(y)dy,$$

then we can rewrite Corollary 2 as follows:

Corollary 4.1. *The following statements (i) and (ii) are valid and equivalent: (i) For any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, we have the following inequality:*

$$\|T^{(2)}f\|_{p,\phi^{1-p}} < K_{\lambda}(\sigma)\|f\|_{p,\varphi}. \quad (4.5)$$

(ii) *For any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\phi}(\mathbf{R})$, satisfying $\|f\|_{p,\varphi} > 0$, and $\|g\|_{q,\phi} > 0$, we have the following inequality:*

$$(T^{(2)}f, g) < K_{\lambda}(\sigma)\|f\|_{p,\varphi}\|g\|_{q,\phi}. \quad (4.6)$$

Moreover, the constant factor $K_{\lambda}(\sigma)$ in (4.5) and (4.6) is the best possible, namely,

$$\|T^{(2)}\| = K_{\lambda}(\sigma).$$

5 Conclusion

In this paper, by means of the technique of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane similar to (1.2) are obtained in Theorem 3.1. The constant factor related to the beta function is proved to be the best possible in Theorem 3.2. As applications, the cases of the homogeneous kernel in Corollary 3.1-3.3, the operator expressions in Theorem 4.1 and Corollary 4.2 are considered. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing Interests

The authors have declared that they have no competing interests.

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Secure Information Sharing System

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Abstract

Privacy emerged as a hot issue again, as the General Data Protection Regulation (GDPR) of EU has become enforceable since May 25, 2018. This paper deals with the problem of health information sharing on a website securely and with preserving privacy. In the context of patient networks (such as ‘PatientsLikeMe’ or ‘USA Patient Network’), we propose the model Secure Information Sharing System (SISS) with the main method of group key cryptosystem. SISS addresses important problems of group key systems. (1) The new developed equations for encryption and decryption can eliminate the rekeying and redistribution process for every membership-change of the group, keeping the security requirements. (2) The new 3D Stereoscopic Image Mobile Security Technology with AR (augmented reality) solves the problem of conspiracy by group members. (3) SISS uses the reversed one-way hash chain to guarantee forward secrecy and backward accessibility (security requirements for information sharing in a group). We conduct a security analysis of SISS according to group information sharing secrecy and an experiment on its performance. Consequently, although current IT paradigm is changing to be more and more ‘complicated’, ‘overlapped’, and ‘virtualized’, SISS makes it possible to securely share sensitive information from collaborative work.

Keywords: Health information sharing; group key; reversed hash key chain; VR/AR; group member indicator.

1 Introduction

This work is about the problem of information sharing on a website which has some groups with members. The Secure Information Sharing System (SISS) is modeled with a group key cryptosystem under the application of patients’ network, which makes it possible to share information with the permitted members.

1.1 Problem identification

Patients around the world want to connect to people who are suffering from the same symptoms and try to find the best treatments. These days, there are some online patient websites for health information sharing such as “PatientsLikeMe” [1] or “USA Patient Network” [2], where patients talk about their symptoms and successful (or failed) experiences. Researchers can also discover new and better solutions based on the patient- contributed data.

The problem is that these kinds of health data may be sensitive and private information. Therefore, patients want the sensitive health data to be protected and managed with safety, and the data to be revealed to only limited persons. That is, the individuals’ right to privacy and control over the circulation of their information [3]. In particular, privacy emerged as a hot issue again, as the General Data Protection Regulation (GDPR) of EU has become enforceable since May 25, 2018.

One of the substantial ways to share patients’ information and to protect privacy is a group key management system. However, a group key system has some peculiar characteristics; the group key should be updated whenever members leave or join the group. This is called rekeying and redistribution. These processes need a complicated and high level of security. Another feature that differentiates an information sharing system from other general group key management systems is that leaving members cannot access the group’s information

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anymore, but joining members can access all the previous group's information to get more information for better treatment. Moreover, current members may conspire with leaving members or other people by revealing their group key [4].

In this work, the proposed model SISS (Secure Information Sharing System) addresses the above problems in the context of online patient networks such as PatientsLikeMe or USA Patient Network. The main methods are group key management system, including the newly developed encryption/decryption algorithm and reversed hash key chain and 3D Stereoscopic Image Mobile Security Technology (augmented reality technique). Namely, that is the secure version for the websites which can make people with the same symptoms share their health information safely and securely.

1.2 Main goals

The SISS's principle to manage users' information with safety is that general information should be presented in plaintexts but sensitive information should be managed in ciphertexts. The solution for sharing ciphertexts with users is group key management system. The following is the main goals of SISS's group key system.

There should be No Processes of Rekeying and Re-distribution. Whenever membership-changes (users leaving or joining groups) happen in the general group key system, the system should generate a new group key and distribute the new key to all members very quickly [5-12]. This process needs more complicated and secure level of techniques, so 'rekeying and redistribution' falls under the hard problem in group key systems. To solve this problem, we develop new equations of the encryption and decryption with the group key. Therefore, SISS has no process of rekeying and redistribution for membership-changes. To the best of my knowledge, this is the first trial to eliminate the process of rekeying and redistribution keeping the security of the group key.

SISS Guarantees Forward Secrecy and Backward Accessibility [13-22]. The representative security requirements for general group key systems are forward secrecy and backward secrecy. However, the security requirements of SISS to share information with group members are forward secrecy and backward accessibility, not backward secrecy. The reversed hash chain can guarantee the security properties of SISS [23-32].

SISS is Collusion-Resistant with New Technology. In every group member system, one of the important security problems is the potential for conspiracy between users and illegal members [33-46]. As a solution, SISS proposes a new concept of 3D Stereoscopic Image Mobile Security Technology using VR/AR technique.

1.3 Key idea and achievements

The main methods and contributions are listed in detail as follows:

- (1) The group key system for encryption and decryption: Only valid users can share the secret information with their group keys and pseudonyms. Others (invalid users) cannot know the contents and ownership for the information: 'what about' and 'whose information'.
- (2) SISS addresses the security problem with reversed hash chain and 3D Stereoscopic Image Mobile Security Technology. And, the newly developed encryption and decryption algorithms are used for efficiency. It is because the information sharing has been and will be highly increased in the networked collaborative computing environments.
- (2.1) Equations for Encryption and Decryption: SISS has no need for rekeying and redistribution for membership- changes. The principle of group key generation for each member is that a fixed master group key is assigned to each group, and random numbers for each user and every session are newly generated by applying random numbers to hash function respectfully, reversedly, and repeatedly s times. Thereafter, each random number and the master group key are combined according to the developed equation algorithm. Then, a total of five subgroup keys are generated as each member's group session key for every session. Hence, every member has completely different group keys for each session and each member because of generating different random number each time. However, one of the most important things is that the results after calculation of all other group keys for the developed equation (for encryption and decryption) are the same as the result of master group key for encryption and

decryption, which makes it possible for SISS not to have rekeying and redistribution processes whenever there are membership- changes.

- (2.2) 3D Stereoscopic Image Mobile Security Technology: It is a new concept of a security solution using VR/AR techniques against conspiracy. The combination of human's facial expressions and gestures which are identified at the registration time of a legitimate group member (LGM) should be rendered as a 3D image in the login process to authenticate a legitimate user. Consequently, the rendering of users' own facial expressions and gestures can prohibit conspiracy between users and invalid persons.
- (2.3) Reversed Hash Chain: SISS should guarantee forward secrecy and backward accessibility, along with group key secrecy [47], which are security requirements in group information sharing system. In SISS, every member's group key is generated based on reversed one-way hash chain. Due to one-way properties of reversed hash function [48], a leaving member cannot know the next group key (forward secrecy) but a joining member can know all the previous keys and information (backward accessibility). Therefore, SISS is suitable for secret collaborative work and sensitive information sharing among group members.
- (2.4) All Different Random Numbers for Each Member and Each Session: In SISS system, every member's group session key looks like private key because their group key has completely different values with different random numbers for each member. Nevertheless, the key plays a role of a group key, with which every group member can share their sensitive health information with other members in a group.
- (3) Stronger authentication for login: The proposed model uses LGM (legitimate group member) for stronger authentication. Moreover, the authentication processes are mutual, so that the proposed model is secure against spoofing or masquerading attack.
- (4) Scalability to other group project systems and mobile phone applicability: SISS is scalable to other group project systems on websites. Although application scenario is about patient networks on the web, SISS is extendable to other secure group projects. Furthermore, it is applicable to even LGM of mobile phones, because the next smartphone will feature a front-facing 3D laser scanner for facial recognition, which was the expectation of upcoming iPhone 8 in early 2018 [49] (but Apple released it without 3D laser scanner in late 2017).
- (5) Privacy preserving system: SISS can meet the privacy requirements: pseudonymity (partial anonymity), unlinkability, and unobservability.
- (6) Blinding: In every flow, SISS uses newly generated random numbers. This masking method does not allow an attacker to know or to guess real contents correctly.

2 Model

2.1 Entities and application scenario

SISS has three parties: users, SM (security manager), and SISS server. SM (security manager) is a kind of a client, which is granted a special role of a security manager. SM is assumed as a TTP (trusted third party) and it is located in front of the SISS server. SM controls group key and key-related information, all sensitive information, and all other events with powerful computational and storage abilities. Fig. 1 shows the system configuration of SISS.

The main participants of SISS are group members who want to get help through information sharing. The information scope is health conditions and patient profile. Mostly, the health information could be shared but some secret personal data in patient profile should be revealed to the allowed people only.

With the mobile devices or PC, Secure Information Sharing System (SISS) is constructed for online website patients with any disease all over the world. The needs are as follows: Many patients want to get in such kind of patient networks and to be helped more easily and securely. Each member uploads his/her conditions or information to his/her personal ciphertext pages (for sensitive data) or plaintextpages (for public data).

One more important thing is that the augmented reality(AR) technique of 3D image rendering is applied to external devices. The rendered image is selected from the contents-list consisting of the randomly repeated and rearranged human's facial expressions and gestures. As the first process, every user should register at SM; thereafter they should get through the authentication processes every session and then they start some actions. When some sensitive information is shared with other patients (it means that the ciphertext pages are generated), we know the page is encrypted by the group's encryption key. Only the legitimate users (who registered at SM and kept the information given by SM at their devices for authentication) can pass authentication processes and know the sharing information. In the last step of the authentication, 3-dimensional image is rendered. This image can be called legitimate group member.

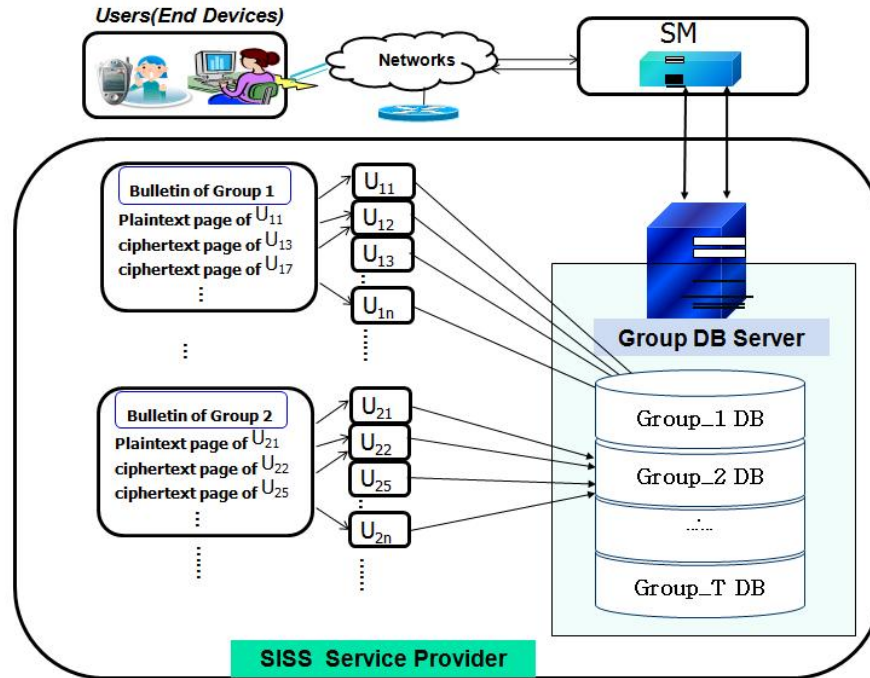


Fig. 1. The configuration of SISS

2.2 Notations

The notations for SISS are as follows.

- SM: Security Manager
- K_G : The group keys set of group G
- G_i : The t -th group
- T : The total number of groups in a SISS server
- mK_G : The master group key of group G
- $CP_{i,t}^j$: The j -th ciphertext-page of a member i in a group t
- n : The total number of group G 's members
- sn : Session number
- m : The total number of ciphertext-page CP
- $U_{i,t}^j$: A member i of the t -th group in the j -th session
- $K_{G_i,t}^j$: A group session key for each member i of t -th group G_i in the j -th session
- $sK_{i,1}^j, sK_{i,2}^j, sK_{i,3}^j, sK_{i,4}^j, sK_{i,5}^j$: Five subkeys for a group session key $K_{G_i,t}^j$ of a member i in group G
- α_i^j : Random number of member i in the j -th session,
- p_i^j : Pseudonym of member i in the j -th session

- $h(\cdot)$: Hash function
- $f(\cdot)$: Pseudorandom function
- S_i^j : Stereoscopic image information for 3D real model of member i in the j -th session
- R_{S_i} : A rendered image of S_i
- E : Encryption function, D : Decryption function
- M : A message

2.3 Algorithms for SISS model

A SISS model consists of the following eight algorithms.

- (1) **SysPrm(I^k)**: Parameter Generation algorithm *SysPrm* takes as an input a security parameter k , and produces a system parameter λ .
- (2) **KeyGen(λ)**: Taken as an input λ , Key Generation algorithm *KeyGen* produces group session keys set RH , group member keys set K , and member pseudonym keys set PD .
- (3) **InfGenStr(λ, RH, K, P)**: Taken as an input λ , Information Generation and Storage algorithm *InfGenStr* produces GMI (group member indicator) and other information for authentication.
- (4) **Qr(RH, K, P)**: Given RH, K, P , Query algorithm produces Query Value Qv .
- (5) **Vr_Qr(Qv)**: Given Qv , Verification algorithm verifies Qv , and Query algorithm produces Query Value Qv .
- (6) **Vr(Qv)**: Given Qv , Verification algorithm verifies Qv .
- (7) **EncUp(M)**: This algorithm encrypts and uploads message M .
- (8) **DnDec(M)**: This algorithm downloads and decrypts message M .

2.4 Security building blocks

Definition 1. One Way Hash Key Chain: It is generated by selecting the last value at random and applying a one-way hash function h repeatedly. The initially chosen value is the last value of the key chain. The followings are two properties of a one-way hash chain.

•**Property 1:** Anybody can deduce that an earlier value k_i really belongs to the one-way key chain by using the later value k_j of the chain and by checking $h^{-i}(k_j)$ which equals k_i with the later value k_j .

•**Property 2:** Given the latest released value k_i of one-way key chain, an adversary cannot find a later value k_j such that $h^{-i}(k_j)$ equals k_i . Even when value k_{i+1} is released, the second pre-image collision resistant property prevents an adversary from finding k'_{i+1} different from k_{i+1} such that $h(k_{i+1})$ equals k_i [48].

Remark: We say that the Property 1 of One-Way Hash Key Chain as ‘backward accessibility’ and the Property 2 as ‘forward security’.

Definition 2. PRF(Pseudo Random Function) We say that ‘ $F: K_f \times X \rightarrow Y$ is (t, q, e) -secure pseudorandom function’ if every oracle algorithm A making at most q oracle queries and with running time at most t has advantage $Adv_A < e$. The advantage is defined as $Adv_A = |Pr[A^{F^k} = 1] - Pr[A^R = 1]|$ where R represents a random function selected uniformly from the set of all maps from X to Y , in which the probabilities are taken over the choice of k and R [50].

Definition 3. PRGG_r (Pseudo Random Generator): We say that ‘ $G_r: K_{Gr} \rightarrow S$ is a (t, e) -secure pseudorandom generator’ if every algorithm A with running time at most t has advantage $Adv_A < e$. The advantage is defined as $Adv_A = |Pr[A(G_r(U_{K_{Gr}})) = 1] - Pr[A(U_S) = 1]|$. Where $U_{K_{Gr}}$, U_S are random variables distributed uniformly on K_{Gr} and S [50].

Definition 4. DDH (Decisional Diffie-Hellman): Let G be a group of prime order q and g a generator of G . The DDH problem is to distinguish between triplets of the form (g^a, g^b, g^{ab}) and (g^a, g^b, g^c) , where a, b, c are random elements of $\{1, \dots, q-1\}$.

The following experiment with a polynomial time adversary A : Flip a coin δ to get 0 or 1, if $\delta = 1$, set $c = ab$, else choose c at random. The DDH problem is said to be hard if for any polynomial time adversary A , $|\Pr(A(G, g^a, g^b, g^c) = \delta) - 1/2|$ is negligible.

Definition 5. DDH (Decisional Diffie-Hellman) The leaving member $U_l \in G_l$ ($1 \leq l \leq n$, $1 \leq l' \leq s$) colluding with the members in the after sessions $U_{l'}^j \in G_j$ ($1 \leq i' \leq n$, $j \geq l'$, $i \neq i'$) cannot recover $K_{G_i}^j$ even knowing $\{\alpha_i^j, K_{G_i}^j\}$ and $\{\alpha_{i'}^j, K_{G_{i'}}^j\}$ [51], where G is a group G at l -th session.

2.5 Security game ICR-IS (Indistinguishability of ciphertexts from randombit strings in information sharing)

•**Setup.** The challenger C creates a board set B of m pages $\{CP_{it}\} \{B_t\}$ ($1 \leq i \leq n$, $1 \leq t \leq T$) and gives this to the adversary A . A chooses a polynomial number of subsets from B . This collection of subsets is called B^* , C runs algorithm $SysPrm$, $KeyGen$, $InfGenStr$ and encrypts each subset running algorithm Qr , $VrQr$, Vr , $EndUp$. Finally, C sends A all ciphertexts with their associated subsets.

•**Queries.** A may request the encryption $EndUp(B^*)$ of any B , and any verification Vr .

•**Challenge.** A choose a B_0 and its subsets such that none of the algorithms (Qr , $VrQr$, Vr) given in the step Queries distinguishes B_0 from $B_l = Rand(B_0)$. The challenger C chooses a random bit b and gives $EndUp(B_b)$ to A . A again asks for encrypted pages and their verifications with the restriction that A may not ask for the algorithm that distinguishes B_0 from B_l . The total number of ciphertext and verification is in k .

•**Response.** A outputs $b_A \in \{0,1\}$. If $b_A = b$, A is successful. In security game ICR-IS, adversary's advantage is defined as $Adv_A(1^k) = |\Pr[b_A = b] - 1/2|$.

2.6 Legitimate Group Member (LGM)

Every user(member) registers at SM with the contents-list which is the combination set for gestures and facial expressions of the user. All the gestures and facial expressions are randomly repeated and rearranged in the contents-list. Then, the user keeps the contents-list in their device for later authentication. $S_{t,i}^j$ is put as the stereoscopic image information for the gesture and facial expression of the member i of group t at the j -th session. Every session, SM selects one of the combinations of gestures and facial expressions from the contents-list and challenges the member of the group. Then, the member renders his own gesture and facial expression for $S_{t,i}^j$.

3 Construction of SISS Model

In this section, SISS is constructed by using the eight algorithms described before. This SISS model is divided largely into four processes; System Setting, Registration, Authentication for Log-In, and Action. The details are addressed in Section 3.1.

3.1 System setting

1) $SysPrm(1^k)$ Construction

- Input; k : a security parameter.
- Output; $\lambda = \{f(\cdot), h(\cdot), G, g, G_r, m, j, i, C, D\}$: system parameters' set.

$f: \{0,1\}^k \times \{0,1\}^* \rightarrow \{0,1\}^k$ is a pseudorandom function and $h: \{0,1\}^* \rightarrow \{0,1\}^k$ is an one way hash function. G_r is a pseudorandom generator. G is a group of order q which is a large prime and g is a generator of a group G . m is the number of group G 's members, j is the session number, and i is each member of group G . E and D are encryption and decryption function.

3.2 Registration

The Registration Process consists of two algorithms; KeyGen(λ), InfGenStr(λ, RH, K, P).

1) KeyGen(λ) Construction

- Input; λ .
- Output; RH(group session random numbers set),
 $K(\text{groupmemberskeysset}), P(\text{pseudonymkeysset})$.

(1) Group Session Random Numbers RH: Reversed One-Way Hash Chain.

It is assumed that the total number of sessions is s . For every member i , each different random number α_i^s ($1 \leq i \leq n$) is generated for the last session. Here, each α_i^s is applied to one-way hash function $(s-1)$ times repeatedly to generate all sessions' random numbers and, respectively, for each use as follows.

$$\begin{aligned} \alpha_i^s &= (\text{randomly generated}), \\ h(\alpha_i^s) &= \alpha_i^{s-1}, \\ h(\alpha_i^{s-1}) &= \alpha_i^{s-2} = h^2(\alpha_i^s), \\ h(\alpha_i^{s-2}) &= \alpha_i^{s-3} = h^3(\alpha_i^s), \\ &\dots\dots\dots \\ h(\alpha_i^4) &= \alpha_i^3 = h^{s-3}(\alpha_i^s), \\ h(\alpha_i^3) &= \alpha_i^2 = h^{s-2}(\alpha_i^s), \\ h(\alpha_i^2) &= \alpha_i^1 = h^{s-1}(\alpha_i^s). \end{aligned}$$

Therefore, the first session's random number of member i is α_i^1 and the t -th session's random number of member i is α_i^t : $h(\alpha_i^{t+1}) = \alpha_i^t = h^{s-t}(\alpha_i^s)$.

With these different random numbers, we can make all different group keys for each member and each session, respectively.

One-way hash function $h(\cdot)$ plays the important role in group information sharing. One-Way hash chain is generated by randomly selecting the last value, which is repeatedly applied to one-way hash function $h(\cdot)$. The initially selected value is the last value of the hash chain. One-way hash chain has two properties as mentioned in Definition 1 in Section 2.4. Therefore, the two properties make it possible that a leaving member cannot compute new keys after leaving the group and any newly joining member can obtain all previous keys and information through applying the current key to hash function $h(\cdot)$ repeatedly.

(2) Group Keys Set K

It is assumed that there are ' n ' members of the group ' G ', and the group session keys for each member i of group G are $\{K_G^j\}$ ($1 \leq i \leq n, 1 \leq j \leq s$). Here, j is a session number and s is the last session. The each member i 's group key K_G^j consists of totally five subkeys; $sK_{i,1}^j, sK_{i,2}^j, sK_{i,3}^j, sK_{i,4}^j, sK_{i,5}^j$. SM selects the master group key mK_g of group $G \in \{G_i\}$, $t \geq 1$ and generates a random number to blind the master key in five subkeys, which is the way to construct a group member's session key and, therefore the last session group key K_G^s of user i .

$$\begin{aligned} sK_{i,1}^s &= h(mK_g)\alpha_i^s, \\ sK_{i,2}^s &= h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_i^s), \\ sK_{i,3}^s &= g^{f_{mK_g}(mK_g)}, \\ sK_{i,4}^s &= -(h(mK_g) + \alpha_i^s), \\ sK_{i,5}^s &= f_{mK_g}(mK_g)\alpha_i^s. \end{aligned}$$

The generation principle of group keys is that every different random number for each member and each session ($n \times s$ random numbers) is combined to the master group key mK_g . The notations of section 2.2 show the random numbers and group keys for each member and each session which belong to the group G. The group G is one of the groups $\{G_t\}$, $t \geq 1$.

(3) Group Members' Pseudonym Keys Set P: Reversed One-Way Hash Chain.

For stronger security and privacy, SISS uses each member's pseudonyms, which are generated with the reversed one-way hash chain in the same way as group session keys. Thus, each member has also s pseudonyms which are denoted as p_i^j (for each member i , $1 \leq j \leq s$).

$$\begin{aligned}
 & p_i^s, \text{ (randomly generated),} \\
 & h(p_i^s) = p_i^{s-1}, \\
 & h(p_i^{s-1}) = p_i^{s-2} = h^2(p_i^s), \\
 & h(p_i^{s-2}) = p_i^{s-3} = h^3(p_i^s), \\
 & \dots\dots\dots \\
 & h(p_i^4) = p_i^3 = h^{s-3}(p_i^s), \\
 & h(p_i^3) = p_i^2 = h^{s-2}(p_i^s), \\
 & h(p_i^2) = p_i^1 = h^{s-1}(p_i^s)
 \end{aligned}$$

Table 1. Random number and group key

		Users		
		U_1	U_2	$U_{3..}$
Session1	Random number	α_1^1	α_2^1	...
	Group key	$K_{G_1}^1;$	$K_{G_2}^1;$...
		$sK_{1,1}^1 = h(mK_g)\alpha_1^1$	$sK_{2,1}^1 = h(mK_g)\alpha_2^1$...
		$sK_{1,2}^1 = h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_1^1)$	$sK_{2,2}^1 = h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_2^1)$...
		$sK_{1,3}^1 = g^{f_{mK_g}(mK_g)}$	$sK_{2,3}^1 = g^{f_{mK_g}(mK_g)}$...
		$sK_{1,4}^1 = -(h(mK_g) + \alpha_1^1)$	$sK_{2,4}^1 = -(h(mK_g) + \alpha_2^1)$...
		$sK_{1,5}^1 = f_{mK_g}(mK_g)\alpha_1^1$	$sK_{2,5}^1 = f_{mK_g}(mK_g)\alpha_2^1$...
Session2	Random number	α_1^2	α_2^2	...
	Group key	$K_{G_1}^2;$	$K_{G_2}^2;$...
		$sK_{1,1}^2 = h(mK_g)\alpha_1^2$	$sK_{2,1}^2 = h(mK_g)\alpha_2^2$...
		$sK_{1,2}^2 = h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_1^2)$	$sK_{2,2}^2 = h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_2^2)$...
		$sK_{1,3}^2 = g^{f_{mK_g}(mK_g)}$	$sK_{2,3}^2 = g^{f_{mK_g}(mK_g)}$...
		$sK_{1,4}^2 = -(h(mK_g) + \alpha_1^2)$	$sK_{2,4}^2 = -(h(mK_g) + \alpha_2^2)$...
		$sK_{1,5}^2 = f_{mK_g}(mK_g)\alpha_1^2$	$sK_{2,5}^2 = f_{mK_g}(mK_g)\alpha_2^2$...
Session3

2) InfGe(λ) Construction

- Input: λ : system parameters' set.
- Output: $S_{t,i}^j, h(E_{K_{G_i}^j}(p_i^j || S)), \{h(E_{K_{G_i}^j}(p_i^j))\}, p_i^1, K_{G_i}^1, \alpha_i^s, \{h(p_i^j), p_i^j\}, (1 \leq j \leq s)$

At the registration process, every user is given some information from SM and stores them in one's own device such as in smartphone or PC; $S_{t,i}^j, p_i^1, K_{G_i}^1, \{h(E_{K_{G_i}^j}(p_i^j))\}, h(E_{K_{G_i}^1}(p_i^1 || S)), (1 \leq i \leq n, 1 \leq j \leq s)$. SM also stores some information for each member i ; $\alpha_i^s, \{h(p_i^j), p_i^j\}, mK_g, S_{t,i}^j, (1 \leq j \leq s)$.

As such, the output of $InfGe(\lambda)$ is the values created by the SM and each user during the registration process, which means the values are stored in advance for later use in the authentication process.

Additional Explanation for Encryption (E) and Decryption (D) with Group Keys. Here, $E_{K_{G_i}^j}(M)$ means ciphertext C with group members' group key ' $K_G = \{K_{G_i}^j\}$ ' for a message M. The encryption with the master key mK_g is assumed $C = E_{mK_g}(M) = g^{h(mK_g)f(mK_g)}M$.

For simplicity, we put $sK_{i,1}^j, sK_{i,2}^j, sK_{i,3}^j, sK_{i,4}^j, sK_{i,5}^j$ as K_1, K_2, K_3, K_4, K_5 , and $f_{mK_g}(mK_g)$ as $f(mK_g)$. Then, the encryption with each member's group key $\{K_{G_i}^j\}$, for example, in the last session (i.e., $j = s, K_{G_i}^s$) is as follows:

$$\begin{aligned} C &= E_{mK_g}(M) = K_3^{K_1} g^{K_2} M \\ &= (g^{f(mK_g)})^{h(mK_g)\alpha_i^s} g^{h(mK_g)f(mK_g)(1-\alpha_i^s)} M \\ &= g^{h(mK_g)f(mK_g)} M \end{aligned}$$

The decryption method with the group master key ' mK_g ' is $D = C \cdot g^{-h(mK_g)f(mK_g)} = M$. Then, the decryption method with the each member's group key $K_{G_i}^s$ in the last session is as follows;

$$\begin{aligned} D &= C \cdot K_3^{K_4} g^{K_5} \\ &= g^{h(mK_g)f(mK_g)} M \cdot (g^{f(mK_g)})^{-h(mK_g+\alpha_i^s)} \cdot g^{f(mK_g)\alpha_i^s} \\ &= g^{h(mK_g)f(mK_g)-h(mK_g)f(mK_g)-f(mK_g)\alpha_i^s+f(mK_g)\alpha_i^s} \cdot M \\ &= M \end{aligned}$$

We can check whether the result of encryption/decryption with the master group key ' mK_g ' is the same as anything of each member's group key $K_{G_i}^j = \{sK_{i,1}^j, sK_{i,2}^j, sK_{i,3}^j, sK_{i,4}^j, sK_{i,5}^j\}$ ($1 \leq i \leq n, 1 \leq j \leq s$). Because of the properties of this developed encryption and decryption algorithms, SISS has no need for rekeying processes whenever membership-changes happen.

3.3 Login by authentication

The login process consists of four algorithms: $QrU1(RH, K, P)$, $VrSM1_QrSM2(QVU1)$, $VrU2_QrU3(QVSM2)$, and $VrSM3(R_{S_i^1})$.

1) $QrU1(RH, K, P)$ Construction

A member i makes login-request to SM with the stored information.

- Input; RH, K, P
- Output; $QvU1$ (1st querying value of a user)

- (1) Compute: $f_{p_i^1}(K_{G_i}^1), h(p_i^1)$; With the stored value $p_i^1, K_{G_i}^1$, a member i computes $f_{p_i^1}(K_{G_i}^1), h(p_i^1)$.
- (2) A member queries SM with $QVU1$
 $QvU1 = 1(sn), h(p_i^1), f_{p_i^1}(K_{G_i}^1), h(E_{K_{G_i}^1}(p_i^1 || S_i^1))$

Here, $h(E_{K_{G_i}^1}(p_i^1 || pS_i^1))$ is also the stored value at registration time. Because $K_{G_i}^1$ is the member i 's group key in the first session, $E_{K_{G_i}^1}(p_i^1 || S_i^1)$ means

$$\begin{aligned} C &= E_{K_{G_i}^1}(p_i^1 || S_i^1) = K_3^{K_1} g^{K_2} (p_i^1 || S_i^1) \\ &= (g^{f(mK_g)})^{h(mK_g)\alpha_i^1} g^{h(mK_g)f(mK_g)(1-\alpha_i^1)} (p_i^1 || S_i^1) \\ &= g^{h(mK_g)f(mK_g)} (p_i^1 || S_i^1) \end{aligned}$$

Here, for simplicity, K_1, K_2, K_3 , are denoted as the member i 's subkeys for its group key $K_{G_i}^1$ in the first session. K_4, K_5 are also the subkeys (for decryption) of $K_{G_i}^1$

2) VrSM1_QrSM2(QvU1) Construction

The SM verifies the login-request of member i and sends the next session information, the group key and the pseudorandom number, to member i .

- Input; Q_vU1
- Output; Q_vSM2 (2nd querying value of SM).

- (1) Find: α_i^s, p_i^1 . SM checks $1(sn), h(p_i^1)$ and finds the corresponding values α_i^s, p_i^1 from its storage.
- (2) SMdecrypts with p_i^1 : $D(f_{p_i^1}(K_{G_i}^1)) = K_{G_i}^1$
- (3) Compute and Verify: $h^{s-1}(\alpha_i^s) = \alpha_i^{1'}$, $K_{G_i}^{1'} = K_{G_i}^1$. For the found value α_i^s , SM applies α_i^s to hash function repeatedly, up to $(s-1)$ times. If he obtains the result $\alpha_i^{1'}$, then SM computes $K_{G_i}^{1'}$;

$$\begin{aligned} K_1^{1'} &= h(mK_g)\alpha_i^{1'}, \\ K_2^{1'} &= h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_i^{1'}) \\ K_3^{1'} &= g^{f_{mK_g}(mK_g)} \\ K_4^{1'} &= -(h(mK_g) + \alpha_i^{1'}), \\ K_5^{1'} &= f_{mK_g}(mK_g)\alpha_i^{1'}. \end{aligned}$$

Then, SM verifies $K_{G_i}^{1'} = K_{G_i}^1$ or not.

- (4) Compute and Verify with $K_{G_i}^1$: $h(E_{K_{G_i}^1}(p_i^1 || S_i^1))' = h(E_{K_{G_i}^1}(p_i^1 || S_i^1))$. Here, $E_{K_{G_i}^1}(p_i^1 || S_i^1)$ is the stored value at the registration time and the encryption method is the same as the above.
- (5) Compute: $\alpha_i^2, K_{G_i}^2$. SM applies α_i^s to hash function $(s-2)$ times and then computes $K_{G_i}^2$;

$$\begin{aligned} K_1^2 &= h(mK_g)\alpha_i^2, \\ K_2^2 &= h(mK_g)f_{mK_g}(mK_g)(1 - \alpha_i^2) \\ K_3^2 &= g^{f_{mK_g}(mK_g)} \\ K_4^2 &= -(h(mK_g) + \alpha_i^2) \\ K_5^2 &= f_{mK_g}(mK_g)\alpha_i^2 \end{aligned}$$

- (6) Compute and Query with Q_vSM2 : $Q_vSM2 = f_{p_i^1}(K_{G_i}^2, p_i^2)$, $f_{p_i^2}(p_i^1 || S_i^1)$, where p_i^2 is also the stored value.

3) VrU2_QrU3(QvSM2) Construction

- Input; Q_vSM2
- Output; $R_{S_i^j}$ (rendering of S_i^j)

- (1) Decrypt: i decrypts $D(f_{p_i^1}(K_{G_i}^2, p_i^2)) = K_{G_i}^{2'}$, $p_i^{2'}$ with the value p_i^1 .
- (2) Compute and Verify: $h(E_{K_{G_i}^2}(p_i^2))' = h(E_{K_{G_i}^2}(p_i^2))$, $h(p_i^{2'}) = p_i^1$. With the decrypted values $K_{G_i}^{2'}$, $p_i^{2'}$, the group member i computes $h(E_{K_{G_i}^2}(p_i^2))'$ and verifies if this is the same as $h(E_{K_{G_i}^2}(p_i^2))$. Then, i hashes the value $p_i^{2'}$ and verifies $h(p_i^{2'}) = p_i^1$. If the verifications are successful, $K_{G_i}^{2'}$, $p_i^{2'}$ become $K_{G_i}^2$, p_i^2 .
- (3) Decrypt with p_i^2 : $D(f_{p_i^2}(p_i^1 || S_i^1)) = p_i^1 || S_i^1$
- (4) Render and Upload: $R_{S_i^j}$ at a page.

4) VrSM3($R_{S_i^1}$) Construction

SM verifies what the member i has rendered.

- Input; $R_{S_i^1}$
- Output; 1 or 0

(1) Verify: $R_{S_i^j}' = R_{S_j}$ (3D facial expression and gesture).

In this process, a legitimate group member authentication is processed by rendering the decrypted S_i^1 with themember's external device. If SM's verification is successful (return message: 1), the member i can begin to act (login allowed). The action means uploading, downloading, and reading (decryption).

3.4 Action

The Action Process consists of two algorithms: $EncUp(M)$, $DnDec(M)$.

1) $EncUp(M)$ Construction

A member i encrypts and uploads the sharing information.

- Input; M
- Output; C_i^1

(1) Encrypt and Upload M by a member i .

$$C_i^1 = E_{K_{G_i}^1}(M) = K_{i,3}^1 \cdot K_{i,1}^1 \cdot g^{K_{i,2}^1}(M) = g^{h(mK_g)f(mK_g)}(M)$$

2) $DnDec(M)$ Construction

Another member u downloads and decrypts the sharing information.

- Input; C_i^1
- Output; M

(1) Download: C_i^1

Another member u downloads C_i^1 from SISS bulletin (server).

(2) Decrypt C_i^1 :

$$D = C_i^1 \cdot K_{u,3}^1 \cdot K_{u,4}^1 \cdot g^{K_{u,5}^1} = g^{h(mK_g)f(mK_g)} M \cdot (g^{f(mK_g)})^{-h(mK_g+\alpha_u^1)} \cdot g^{f(mK_g)\alpha_u^1} \\ = g^{h(mK_g)f(mK_g)-h(mK_g)f(mK_g)-f(mK_g)\alpha_u^1+f(mK_g)\alpha_u^1} \cdot M = M$$

[The Second Session]. From the second session, most processes are similar to the first session. As the session is changed, the corresponding pseudonym keys and group session keys are also changed. As for the stereoscopic image information S for 3D real model, a member sends the information S^t kept from the first session to SM, and then SM challenges the member with the newly selected information S^{t+1} at (6) of the algorithm $VrSM1QrSM2$. Lastly, the member renders 3D real model $R_{S^{t+1}}$ at his page. Action stage is also similar to the first session.

4 Security Analysis

The security requirements related to group key are as follows:

- (1). *Group Key Secrecy*: It should be computationally impossible that a passive adversary discovers any secret group key.
- (2). *Forward Secrecy*: Any passive adversary with a subset of old group keys cannot discover any subsequent (later) group key.

- (3). *Backward Secrecy*: Any passive adversary with a subset of subsequent group keys cannot discover any preceding (earlier) group key.
- (4). *Key Independence*: Any passive adversary with any subset of group keys cannot discover any other group key [4].

In this work, the term negligible function refers to a function: $N \rightarrow R$ such that for any $c \in N$, there exists $n_c \in N$, such that $(n) < 1/n_c$ for all $n \geq n_c$ [47].

The model SISS satisfies group information sharing secrecy as follows: (1) forward secrecy, (2) backward accessibility, (3) group key secrecy, and (4) collusion resistance.

Theorem 1 (forward secrecy). For any group, an adversary A (including a participant $p \in G_t^j$) cannot know valid group key for $(j+l)$ -th authentication when the adversary A knows group key $K_{G_i^j}$, where $p \in \{G_t^{j+l}\}^c$ ($1 \leq j \leq s, 0 < l \leq s - j, 1 \leq t \leq T$).

Proof. By Property 3 of Definition 1, if the latest released group key is $K_{G_i^j}$, no one can know a later value $K_{G_i^j}$ such that $h^{(l-j)}(K_{G_i^l}) = K_{G_i^j}$. Therefore, the probability that a participant $p \in G_t^j$ can generate valid group keys for the next l -th session is negligible, where $p \in \{G_t^{j+l}\}^c$ ($j < l \leq s$). It means that all leaving group members cannot access any of the next documents of the group anymore.

Theorem 2 (backward accessibility). For any group G_t , an adversary A (including a participant $p \in G_t^j$) cannot generate valid group key for $(j-l)$ -th authentication when the adversary A knows group key $K_{G_i^j}$, where $p \in \{G_t^{j-l}\}^c$ ($0 < l < j$). Namely, all joining members to a group can access all of the previous information of the group.

Proof. By Property 2 of Definition 1, if the latest released group key is $K_{G_i^j}$, anyone can deduce earlier values $K_{G_i^l}$ ($0 < l < j$) by applying the later value $K_{G_i^j}$ to one-way hash key chain like this: $h^{(j-l)}(K_{G_i^j}) = K_{G_i^l}$. Therefore, the probability that a participant $p \in G_t^j$ can generate valid group keys for the earlier l -th session is $1 - (n)$, where $p \in \{G_t^{j-l}\}^c$ ($0 < l < j$). Namely, all members joining a group can access all of the previous information of the group

Theorem 3 (group key secrecy). For any group G_t , when a revelation of group key $K_{G_i^j}$ happens, the probability that an adversary A (including a participant $p \in G_t^j$) can guess correctly the encrypted information message M of group G_t at the j -th session is negligible.

Corollary 1. SISS is semantic secure according to the security game ICR-IS, if DDH is hard and the key material is chosen as described in the algorithm construction.

The cryptographic elements for authentication and whole protocol are PRF (pseudorandom function, e.g., 128 bit-AES), PRG (pseudorandom generator, e.g., middle-square method, Naor-Reingold pseudorandom function, etc.), and hash function (HAS-160), generally known as secure cryptographic function. Through the cooperative processes of these elements, the final encryption is $C = E_{K_{G_i^j}}(M) = g^{h(mK_g)} f(mK_g)(M)$. Hence, we have only to show the security under the condition of ‘DDH is hard’.

Proof (it is proved by contraposition). A is assumed as an adversary that wins the security game ICR-IS with advantage ϵ . We construct an adversary Ω , which uses A as a subroutine and breaks the DDH with non-negligible advantage.

- (i) *Setup.* Algorithm Ω creates m message pages $\{M_{i,t}^j\} \in \{M_t^j\}$ ($1 \leq i \leq n, 1 \leq t \leq T, 1 \leq j \leq m$) and gives this to the adversary A .

A chooses a polynomial number of subsets $\{M_{i,t}^j\} \in \{M_t^j\}$ from messages set M. This collection of subsets is called M^* . A sends them to Ω again. Ω invokes algorithm *SysPrm*, *KeyGen*, *InfGenStr*. After creating all ciphertext pages $\{CP_{i,t}^j\} \in \{CP_i\}$ for M^* , Ω gives them and their associated subsets to A.

Here, let $g^{\delta_{\tau}}$ be a Diffie-Hellman triplet; the challenge is to determine $\gamma = \delta\tau$. Ω guesses a value CP_x for the page CP_y that A will choose in the game ICR-IS, by picking M_x uniformly at random in $\{M_{i,t}^j\}$ ($1 \leq t \leq T$). Ω simulates the algorithm *EndUp* on $\{CP_{i,t}^j\}$ as follows. Ω maps every ciphertext page $\{CP_{i,t}^j\}$ to a random value $\{x_{i,t}^j\}$. For $B = \{B_t\} = \{CP_{i,t}^j\}$, Ω chooses random number γ_t and outputs the following.

$$\begin{aligned} \{B_t\} (1 \leq i \leq n, 1 \leq t \leq T, 1 \leq j \leq m) &= \{CP_{i,1}^j\} \\ &= g^{h(K_1)f(K_1)}(M_{i,1}^j) = \{g^{\gamma_1}(x_{i,1}^j)\} \\ &= \{g^{\delta_1\tau_1}(x_{i,1}^j)\} = \{CP_{i,2}^j\} = \{g^{h(K_2)f(K_2)}(M_{i,2}^j)\} \\ &= \{g^{\gamma_2}(x_{i,2}^j)\} = \{g^{\delta_2\tau_2}(x_{i,2}^j)\} \\ &\dots\dots \\ &= \{CP_{i,T}^j\} = \{g^{h(K_T)f(K_T)}(M_{i,T}^j)\} = \{g^{\gamma_T}(x_{i,T}^j)\} \\ &= \{g^{\delta_T\tau_T}(x_{i,T}^j)\} \end{aligned}$$

(ii) *Queries*. If A queries for the message page $\{M_{i,t}^j\}$, Ω outputs the ciphertext page $\{CP_{i,t}^j\} = \{g^{\delta_t\tau_t}(x_{i,t}^j)\}$.

(iii) *Challenge*. Finally, A selects a challenge page set $B_0 \in M$ at random and generates another page set B_1 from M. Next, A gives B_0, B_1 to Ω . Ω chooses $b \xleftarrow{R} \{0,1\}$ and chooses random number γ_b . Ω returns to A the following ciphertext: In the case of $b=0$, $CP_y = g^{\gamma_0}(x_{i,0}^j)$. If $b = 1$, Ω returns random value in reply to DDH challenge. If $\gamma = \delta\tau$, this is an encryption of $\{CP_x\}$; otherwise it is not. A is again allowed to ask for pages of the Board set with the restriction that A must not make a query to distinguish $\{CP_{i,0}^j\}$ from $\{CP_{i,1}^j\} = \text{rand}(CP_{i,0}^j)$ where $\{CP_{i,0}^j\}$ means a DDH triplet and $\{CP_{i,1}^j\}$ is not a DDH triplet.

(iv) *Response*. A outputs a bit b' . If $b' = b$, Ω guesses that $g^{\delta_{\tau}}$ constitute a DDH triplet. If $b' \neq b$, Ω guesses that $g^{\delta_{\tau}}$ do not constitute a DDH triplet. Since the encryption will be random for the page $\{CP_x\}$ if and only the challenge is not a DDH tuple, Ω solves the DDH challenge with the same advantage that A has in winning security game ICR-IS.

It is shown that Ω can solve the DDH problem ($\gamma = \delta\tau$) with non negligible probability. Accordingly, the advantage of Ω in winning this experiment is as follows.

$$\begin{aligned} Adv_{\Omega} &= Pr[\text{Exp}_{\Omega}^{\text{DDH}} = 1] = Pr[b' = b] \\ &= Pr[b' = b | b = 1] \cdot Pr[b = 1] + Pr[b' = b | b = 0] \cdot Pr[b = 0] \\ &= Pr[b' = b | b = 1] \cdot \frac{1}{2} + Pr[b' = b | b = 0] \cdot \frac{1}{2} \\ &= Pr[b' = 1 | b = 1] \cdot \frac{1}{2} + Pr[b' = 0 | b = 0] \cdot \frac{1}{2} \\ &= Pr[b' = 1 | b = 1] \cdot \frac{1}{2} + (1 - Pr[b' = 1 | b = 0]) \cdot \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2} (Pr[EXP_A^{\text{ICR-IS-1}} = 1] - Pr[EXP_A^{\text{ICR-IS-0}} = 1]) \\ &= \frac{1}{2} + \frac{1}{2} Adv_A^{\text{ICR-IS}} = \frac{1}{2} + \frac{1}{2} \epsilon \end{aligned}$$

Theorem 4 (collusion resistance). For any leaving member $U_i^j \in G^l$ ($1 \leq i \leq n, 1 \leq l < l' \leq s$) including any other adversaries, SISS is ns -collusion-resistant.

Proof. For anyone U_i^j ($1 \leq l < l' \leq s$) colluding with the legitimate member U_i^j , the illegal member cannot compute $K_{G_i}^j$ ($j \geq l$). Although the compromised (illegal) member i knows $\{\alpha_i^l, K_{G_i}^l\}$ and $\{\alpha_i^j, K_{G_i}^j\}$, the illegal member cannot receive $\{\alpha_i^{l+1}, K_{G_i}^{l+1}\}$ of the next $(l + 1)$ -th session. Hence, they cannot compute $K_{G_i}^j$ ($1 \leq i \leq n, 1 \leq l < l' \leq j \leq s$). One more important thing is that the illegal member cannot pass the verifiable process to render the real 3D image from the stereoscopic information S_i^j which consists of the member's own gesture and facial expression. Therefore, the illegal member cannot pass the authentication process of login.

5 Performance Analysis

The main purpose of this paper is to design a prototype scheme for secure patient networks. In addition, we try to apply a new technology like the AR/VR technique of 3D model to the authentication process. However, the performance for whole protocol of SISS largely depends on the network condition. Hence, we experiment the performance of SISS with separate eight parts as follows: (1) the generation time in a server including storage time, (2) the time for a client including data transfer and storage in DB, (3) $QrU1$ of login, (4) $VrSM1QrSM2$ of login, (5) $VrU2QrU3$ of login, (6) $VrSM3$ of login, (7) $EncUp$, and (8) $DnDec$.

Table 2. The specification of a server

Server		
Hardware	CPU	Intel Core i7-4770K 3.5 GHz
	Memory	4 GB
	Disk	200 GB
OS	Ubuntu 14.04 LTS / Linux Kernel 3.19.0	
Development Tool	gcc 4.8	

Table 3. The specification of a client

Client (Nexus 5X)		
Hardware	CPU	Snapdragon 808 1.8 GHz
	Memory	2 GB
	Disk	16 GB
OS	Android 6.0	
Development Tool	eclipse / Android SDK	

5.1 Implementation and experimental environment

The experimental environments of a server and a client are addressed in Tables 2 and 3.

5.2 Cryptographic parameters and library

Cryptographic parameters and libraries are described in Table 4.

Table 4. Cryptographic parameters and libraries

Pseudorandom function f	AES 128 bits
Hash function h	SHA-1
Group parameter	cyclic group
Modulus P	2048 bits
Order q	256 bits
Generator g	2048 bits
Time measurement function in a Server	clock()
Time measurement function in a Client	System.currentTimeMillis()
Crypto Library of a Server	MIRACLE
Crypto Library of a Client	Android OpenSSL & Java BigInteger

5.3 The results of implementation

Table 5 shows the performance of SISS divided into eight parts. The registration process which needs only once to join the website totally takes over 20seconds. Considering that users should generate all their information to use it through all their sessions at this registration process in advance, the estimated time is understandable and applicable to a real world in general. Other processes such as login or actions take much less than 1 second (cf. in the implementation of *VrUQrU3*, the rendering process can be skipped because there is no commercialized tools for rendering until now; we replace $R_{S_i^j}$ with 20-byte 3-dimensional image).

Table 5. The result

Operation		Process time (ms)
Registration (1000 Session)	The generation time in a server including storage time	6138 ms (6.1 s)
	The time for a client including data transfer and storage in DB	14499 ms (14.4 s)
Login	<i>QrU1</i>	16 ms (0.016 s)
	<i>VrSM1_QrSM2</i>	2 ms (0.002 s)
	<i>VrU2_QrU3</i>	56 ms (0.056 s)
	<i>DVrSM3</i>	Skip(comparison between simple values)
<i>EncUp</i>		54 ms (0.054 s)
<i>DnDec</i>		32 ms (0.032 s)

5.4 Comparison with other works

The related works' main goal was only focused on the methods of group key's rekeying, revocation, and redistribution, whereas SISS's main goal is to design the information sharing protocol with safety for application website. Hence, the Storage Overhead and Communication Overhead analyzed in the related works are obviously different from SISS because our proposed group key system is developed to eliminate the processes of rekeying and redistribution that constitute hard and complicated work with heavy overheads. To the best of my knowledge, SISS's group key is the first scheme without rekeying and redistribution; nevertheless it can guarantee the security requirements of group key.

Gou et al.'s paper [51] is the latest work to analyze the performances of current schemes until now; the minimal Storage Overhead is $\log_2 p$ (p : finite field's order) and Communication Overhead is $(n + 2)\log q$ (n : maximum revoked users, j : session, q : multiplicative group's order). As for SISS, Storage and Communication Overheads are $O(0)$. Thus, based on Gou et al.'s work, we only compare and analyze the security performances of group key because the proposed scheme SISS has no process itself for group key's rekeying and redistribution.

Table 6 shows that Staddon et al. and Liu et al.'s schemes can guarantee only forward secrecy, and Rams et al. and Guo et al.'s new scheme can meet all properties of forward secrecy, backward secrecy, and collusion resistance. The number of Revocation Limit is the maximum for Guo et al.'s new scheme and the proposed scheme SISS. And, SISS can guarantee all security properties of forward secrecy, backward accessibility, and collusion resistance, but not backward secrecy.

Table 6. The comparison of group key security performance

Scheme	Revocation limit	Forward secrecy	Backward secrecy	Collusion resistance
Staddon et al.	n	Yes	No	No
Liu et al.	n	Yes	No	No
Guo et al.'s basic scheme	n	Yes	No	No
Guo et al.'s new scheme	ns	Yes	Yes	Yes
SISS	ns	Yes	Backward Accessibility	Yes

6 Discussion

6.1 The differences from other group key structure

(1) The application and aim of our group key are different from the traditional group key. Rather, they are closer to 'keyword search schemes for multiuser setting with group keys', whose security requirements are forward secrecy and backward accessibility; the leaving members should not know the group's documents, and newly joining members should know the previous documents of the group to perform the group's tasks. In the sense of sharing information among group members, we used the term group key.

(2) The formation structure of the group key is completely different. General group key systems make every user share the same group key for the session. However, in SISS, on the basis of the master key, random numbers and other things are combined, where the random number has a different initial value for each user, which is hashed $(s - 1)$ times for s sessions (total number of sessions is s). Finally, each user has different group keys for each session and does not share any key with anyone, except for K_3 only. The important thing is that even members themselves do not know their group's master key because it is masked with random numbers. At registration, s hashed values generated through hash chain are stored in only SM's server. Each user has only $S_{t,i}^j, p_i^1, K_{G_i}^1, \{h(E_{K_{G_i}^j}(p_i^j))\}, h(E_{K_{G_i}^1}(p_i^1|S))$, $(1 \leq i \leq n, 1 \leq j \leq s)$ for the authentic information required at the start of the session as mentioned in the algorithm *InfGenStr*.

(3) The result of encryption/decryption with the group's master key and the result of encryption/decryption with the every member's group key are the same (refer to [Additional Explanation for Encryption (E) and Decryption (D) with Group Keys] in Section 3). This is because the developed equation (algorithm) is designed according to the principle that all random numbers attached before the computation should be removed after the computation. It makes rekeying and redistributing of the group key unnecessary for SISS. In SISS, members can upload only on their web pages, while download can be done on their own web pages and those of other users (valid users). Therefore, members encrypt the information that they want to share with the group key and encrypt the information that they want to be secret with their private keys.

(4) It can be said that the group key renewals for session changes are accomplished in the authentication processes of login for each member. In other words, the group key and pseudonym key for each user's next session are given by the SM at the end of the login process, which serve more as authenticators to pass the login process. If any member does not receive the group key and pseudonym key for the next session from the SM, the value can never be deduced. The reasons are as follows: (1) Group keys have an effect similar to a one-time password because they have completely different values for each member and for each session. (2) The master key and the random number cannot be inferred because of the combined characteristics (safety) between masterkey and random number such as DDH, DLP, and other cryptographic functions. (3) Due to the hash chain's one-wayness, which is the method of random number generation, we never know the random number of the next session, so we do not know the group key value of the next session.

(5) The leave and revocation process of SISS is also different from the general group key because SISS does not have a rekeying and redistributing process. When SM receives the leave request from a member, the SM enters the revocation process, records the member's id in the leave-list, and deletes the user's hash chain and other additional information. Even if a member who has left a session tries to log in with the next session information which is received from the previous session, the member cannot pass the authentication because all information of the user has been removed from SM's server. And the member cannot receive the next session information any longer. In other words, a member can no longer log in to the group if the member leaves the group, so that the member should download all the previous information before requesting leave. The leaving members can never know the next subsequent information, while newly joining members can decrypt all the shareable information encrypted with the group key.

(6) The meaning of a session of SISS is different from other general group key systems that consider the session as the number of membership-changes, as SISS considers the session as the number of logins for each member. If a member has performed a total of s logins, then the member can reconnect to SM and generate a new hash

chain again as he did at the registration time. The total number of sessions, s , can be determined by the policies of the website or by the needs of individual members.

6.2 Legitimate group member

In the last step of the login authentication, 3-dimensional image R_s is rendered. R_s plays a role of a LGM (legitimate group member) which is decided with SM at the registration time. The goal is “improving authentication and security against conspiracy and compromise”. If 3-dimensional image is inefficient in a real world, 2-dimensional image is recommendable.

In 2016, Google’s project ‘Tango’ has been showcased with indoor mapping and VR/AR platform [52]. ‘Tango’ technology enables a mobile device to measure the physical world. Tango-enabled devices (smartphones, tablets) are used to capture the dimensions of physical space to create 3D representations of the real world. ‘Tango’ gives the Android device platform the new ability of spatial perception.

According to JPMorgan analyst Rod Hall [49], Apple expects that iPhone 8 would feature a front-facing 3D laser scanner for facial recognition. It can be also said that the facial recognition will potentially be more secure than Touch ID, and 3D laser scanner could eventually be used for other purposes such as augmented reality. Unfortunately, however, the iPhone 8 released in 2017 did not have the expected function of front-facing 3D laser scanner. Even though the released AR technique of iPhone 8 was different from the 3D laser scanner for facial recognition by Rod Hall [49], we can anticipate the generalized AR technique for the facial recognition in the near future. Therefore, we can say that the proposition of SISS is timely good to apply LGM to the real world keeping abreast of Tango and iPhone’s AR/VR technique of mobile devices.

6.3 Privacy preserving SISS

SISS can meet the privacy requirements as follows:

- (1). Anonymity and Pseudonymity: In SISS, each member uses different pseudonymity for each session. Although perfect anonymity cannot be provided, pseudonymity can be provided instead.
- (2). Unlinkability: Every session, users log in with different pseudonyms (P) and use different encryption keys (each member’s groupkey). Consequently, SISS can achieve unlinkability and similar level of security to ‘One-Time Encryption’.
- (3). Unobservability: All information is encrypted by members’ group keys, which have different values by being masked with the differently generated random numbers for each user and each session [53].

6.4 Mutual authentication

An attacker may try to pretend to be a valid member to log in to the SIS system or masquerade as an SM server to extract users’ information. This property is about spoofing attack.

The authentication between a member and the SM server is accomplished through the query and verification algorithms: Qr , $VrQr$, Vr . Specifically, authentication processes for login consist of $QrU1$, $VrSM1_QrSM2$, $VrU2QrU3$, and $VrSM3$. In $QrU1$, a member queries the SM with Q_vU1 which is the computed values using the stored values at the registration time. Then, in $VrSM1QrSM2$, the SM server verifies the value Q_vU1 with the stored values, too. After the successful verification, the SM server queries the member with Q_vSM2 , which is also computed using the stored values and Q_vU1 . To the last processes $VrSM3$, the member and the SM server authenticate each other using the stored values, respectively.

From a member to the SM server, if the SM server can obtain the corresponding rightly rendered image in the last authentication process, it means that the SM server is the real server to which a member wants to log in and the member is a valid user to be registered in advance.

7 Conclusion

SISS is the proposal for the patients from all over the world who want to get some help and share information through websites such as ‘PatientsLikeMe’ or ‘USA Patient Network’. The proposed model SISS can guarantee

security and privacy for the sensitive health and private information. As for the main method of group key management system, SISS addressed the hard problems of rekeying and redistribution, conspiracy, and backward accessibility with new ideas such as equations for encryption/decryption and LGM. Moreover, SISS is scalable to general group's project applications with safety. Therefore, it is clear that the problem of information sharing and the approaches between collaborative computing and security should be managed as Integrated Security Management (ISM).

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Competing Interests

Author has declared that no competing interests exist.

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Current Research on Significance of Artificial Intelligence and Machine Learning Techniques in Smart Cloud Computing: A Review

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Abstract

Realization of the tremendous features and facilities provided by Cloud Computing by the geniuses in the world of digital marketing increases its demand. As customer satisfaction is the manifest of this ever shining field, balancing its load becomes a major issue. Various heuristic and meta-heuristic algorithms were applied to get optimum solutions. The current era is much attracted with the provisioning of self-manageable, self-learnable, self-healable, and self-configurable smart systems. To get self-manageable Smart Cloud, various Artificial Intelligence and Machine Learning (AI-ML) techniques and algorithms are revived. In this review, recent trend in the utilization of AI-ML techniques, their applied areas, purpose, their merits and demerits are highlighted. These techniques are further categorized as instance-based machine learning algorithms and reinforcement learning techniques based on their ability of learning. Reinforcement learning is preferred when there is no training data set. It leads the system to learn by its own experience itself even in dynamic environment.

Keywords: Cloud computing; load balancing; optimal solution; artificial intelligence and machine learning techniques; instance-based learning; reinforcement learning.

1 Introduction

In today's world, businessmen at various levels have realized the necessary of automated decision making systems to learn their customers' behaviour and lead their business successfully. The automated intelligent system should be capable to analyze the heterogeneous data generated in multiple sources and identify the underlying patterns and knowledge to support decision making. The generated model is trained with training data, and tested with validation data. Further, the model has to analyze the newly arrived data, and identify their pattern or the hidden knowledge. ML algorithms are categorised as supervised, unsupervised, reinforcement, deep learning algorithms, fuzzy logic and evolutionary computations.

Supervised learning algorithms use discrete or continuous quantity of labelled data. It consists of classification and regression methods which can be used for data categorization and prediction. The unsupervised learning methods are used to find the efficient representation of unlabelled data. Clustering and dimension reduction are the two basic unsupervised learning methods. In vehicular wireless network, it supports the formation of risk free communication system [1].

Reinforcement Learning (RL) interacts with the dynamic environment in a trial-and-error manner, and maps the situations and actions based on maximized reward value. Another popular ML technique is the deeper version of Neural Network (NN), known as Deep Learning (DL). It makes the system to learn from the data represented by any other category of ML algorithms. Evolutionary computations such as evolutionary algorithms are classified as Genetic Algorithm (GA), Meta-heuristic Algorithms and Swarm Intelligence Algorithms.

2 Literature Survey

In [2], Hemlata, et.al considered the support of Cloud Computing in the analysis of big data and their concerns during the migration process for load balancing. As per their research results, their proposed algorithm,

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EAMLB, Enhanced Active Monitoring Load Balancing algorithm was performed well than that of Round Robin method which was equated to deep learning.

In [3], Bakul, et al. were applied regression technique to predict VMs load and for queue updation. Based on the comparison of VMs load with the upper and lower threshold values, separate queue of under-loaded and overloaded VMs were formed by the Queue Manager which was considered in further VM allocation.

In [4], Mousa et al. were addressed VMs allocation concerns with classification technique. It classified the users requested VMs based on their CPU and RAM requirement which are collected from user's log files. Its tasks and VMs group mapping improved the performance with reduced response time and jobs rejection.

In [5], Renu Choudhary, et al. was proposed the new load balancing algorithm based on k-means clustering algorithm to cluster the available VMs based on their engaged CPU time and memory. Their approached throttled algorithm had updated and sorted the VMs information based on their throughput value which are used to manage further scheduling.

Hao, et al. in [1], were developed an intelligent vehicular transportation system to support the establishment of smart cities. ML algorithms were used as data-driven approach, and Poisson Regression Tree (PRT) method was used to correlate two tasks in order to predict the communication connectivities and vehicles traffic. DL, Fuzzy Logic refined with RL technique, and k-means clustering mechanism were used to cluster the traffic data and to control network congestion.

In [6], Mohammad, et al. were proposed the development of Aarhus City Smart Traffic IoT application. It was framed with the support of Supply Vector Machine (SVM) algorithm to get traffic information for the entire day. They were also analyzed the usage of various ML algorithms in the development of smart cities.

When the researchers were inspired with the hidden intelligence in swarm and other sort of natural resources, they were realized that their admired knowledge of these resources were suitable for their research problems also to get optimal solutions. Such kinds of algorithms applied in load balancing area of Cloud Computing were analyzed by the authors Dalin, et.al in their survey paper [7].

In [8], Stelios Sotiriadis, et al., were proposed a Self-managed VM Placement (SVMP) algorithm. This model was used real-time resource usage data to train the system. The collected data were applied with classification and regression techniques to define the actual behaviour of PMs as well as VMs. Based on the results, it was known that there was a significant improvement in the placement of VMs.

Mohammadreza, et al., in [9], were surveyed the Cloud load balancing techniques under three different categories, namely, General Algorithms-based, Architectural-based, and Artificial Intelligence-based load balancing mechanisms. Under the AI part of survey, they had analyzed the significance of Ant Colony Optimization (ACO), Bee Colony Algorithm (BCA), and an improved ACO in balancing of peak time load, power consumption reduction.

In [10], XIAOFEI WANG, et al. were considered AI techniques to deal with the technical issues in mobile Heterogeneous Network (HetNets). They were used GA, ACO, and Particle Swarm Compensation (PSC) algorithms to set right the self-configuration HetNets, by improving its self-healing and self-optimization capabilities. Fuzzy Neural Network Optimization based on RL technique was used for autonomous reconfiguration.

Amandeep Kaur, et al. in [11] also surveyed the applicability of AI algorithms in Cloud load balancing process. FUGE, the load balancing algorithm was combined with the knowledge of Fuzzy Logic and GA. It increased the performance of job scheduling round robin method. ACO-VMM algorithm was implemented in VM Manager (VMM) system to improve its migration process. Utilization of BABC, the Binary Artificial Bees Colony algorithm provided flexible ranking strategy for balancing the searching and utilization processes. The comparison report revealed that GA was performed well than other algorithms.

3 Results and Discussion

Thus this proposed review reveals the researchers and industrialists interest in the utilization of AI-ML techniques to enhance their system in interest. Recently, Vehicular Transportation System, mobile

Heterogeneous Networks (HetNet), Cloud Computing, are the major systems which are applied with the ML techniques Classification, Regression, Poison Regression Tree, Supply Vector Machine, Fuzzy Logic, Ant Colony Optimization, Deep Learning, and Reinforcement Learning in different concerns to make them as self-configurable and self-learnable smart systems. In some of the researches [1,7,10,11], the Fuzzy Logic is combined with their chosen heuristic or meta-heuristic algorithm. The review report is summarized in Table 1.

Table 1. A Survey Report – Utilization of various AI techniques in different areas of interest

Ref. No.	AI techniques	Applied area	Purpose	Merits
[2]	Enhanced Active Monitoring Load Balancing (EAMLB) method	Cloud Computing	Migration of VMs	Performed well than Round Robin method
[3]	Queue Manager with Regression Technique	Cloud Computing	Identify the VMs whose load is above upper_threshold or less than lower_threshold value	Improved VM allocation
[4]	Classification Technique	Cloud Computing	Grouping the VMs based on CPU and RAM utilization and Grouping of Users task using log files information	Groupwise mapping of users task and VMs
[5]	K-Means Clustering Algorithm	Cloud Computing	Clustering the VMs based on CPU and memory utilization	Data Center Controller System utilizes that information to update its Hash map index value to support scheduling process
[1]	Poisson Regression Tree, Deep Learning, Reinforcement Learning, Deep Reinforcement Learning, Fuzzy Logic with Reinforcement Learning	Development of Smart Cities	Intelligent Vehicular Transportation System	Prediction of communication connectivities and vehicles traffic
[6]	Supply Vector Machine (SVM) technique	Development of Smart Cities	Provision of Traffic Information	Traffic information is provided for the entire day
[7]	Dynamic Load Balancing with effective Bin Packing and VM Configuration (DLBPR) Fuzzy decision based TOPSIS algorithm Genetic Algorithm Stochastic Hill Climbing algorithm Honey Bee Behavior Inspired Load Balancing (HBB-LB) algorithm Multi Agent (MA) system with the knowledge of Ant to deal with dynamic load balancing issues, and the knowledge of Hybrid Fruitfly was applied with the optimization technique	Cloud Computing	Load balancing processes	Deadline based job scheduler, Chosen of appropriate VM for migration, Proper distribution of load, Improved VM allocation Maximizing the throughput Deal with dynamic load balancing issues optimization technique to get optimum solution with improved performance, resources utilization, and reduction in

Ref. No.	AI techniques	Applied area	Purpose	Merits
			to get optimum solution.	response and execution time
[8]	Classification and Regression Techniques	Cloud Computing	Self-managed placement of VMs	Defining the actual behaviour of PMs and VMs
[9]	Ant Colony Optimization (ACO) Bee Colony Algorithm (BCA) Improved ACO	Cloud Computing	Load balancing processes	Migration of overloaded VMs into under-loaded PMs Shifting of under-loaded PMs load to reduce power consumption
[10]	Genetic Algorithm (GA), Ant Colony Optimization (ACO) Particle Swarm Compensation (PSC) algorithm Fuzzy Neural Network Reinforcement Learning	Mobile Heterogeneous Network (HetNet)	Self-configuration Self-healing and Self-optimization	Planning and placement of resources Reconfiguration with the newly identified resources Centralized cell management scheme Autonomous reconfiguration Optimization method
[11]	Fuzzy Logic and Genetic Algorithm – FUGE Ant Colony Optimization Algorithm Binary Artificial Bees Colony algorithm Genetic Algorithm	Cloud Computing	Load balancing processes	Job scheduling Support VM manager system to handle migration process Balancing searching and utilization processes

The above review depicts that the ability of RL technique, i.e., self-learning from dynamic environment opens a new vision of wisdom in various research areas. The RL, combination of RL with other ML methods [1,10], and DRL [1] are in emerge usage of research interest. In this era, the revolution of Industry 4.0, is in the need of intelligent, self-manageable automated systems everywhere. The auto learning RL method is applicable for Cloud load managing process to make it as smart self-load-management system [2,7,9].

4 Machine Learning Techniques

The system modelled with ML technique is trained with the existing data set. Further, the learning ability of the model is verified with the validation data set. There are numerous categorization of ML techniques exists. In this study, the ML techniques are categorized based on their learning ability, i.e., Instance-Based Learning ML (IBL-ML) algorithms, and Reinforcement Learning (RL) techniques. Consider the Fig. 1.

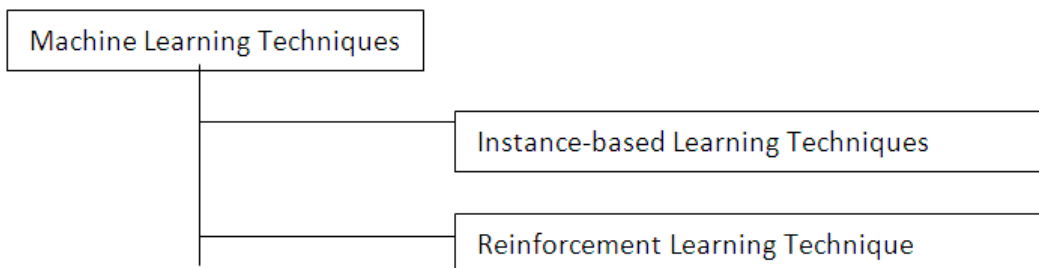


Fig. 1. Categorization of machine learning algorithms

4.1 Instance-based learning techniques

IBL-ML algorithms are also part of the supervised learning category, but it can deal with the fuzzy data set also. With the obtained data, the model is formed either with the classifier or regression method. The model's resultant information are more useful to improve the decision making process. It is known that the trained model will be able to easily predict the relation between the observed data and the system behaviour.

It enforces the selection of suitable prediction algorithm that should be computationally light and obtain good results. It needs efficient training data set to improve the system performance.

4.2 Reinforcement learning techniques

RL enforces the system to learn by its own experience itself. It can handle the dynamic environment of the system. At the beginning, in trial-and-error manner, it interacts with the system. Two main strategies approached by RL are searching the space of behaviours in order to identify the one which performs well in that environment, and defining the statistical function in dynamic programming to estimate the reward value of the action which may be planned to occur in the current state of the environment.

The second strategy can be applied with the self-management approaches to easily adopt with the drifts and environmental changes in the system. The RL learning process identifies the suitable set of policy and action. To estimates the returns, function can be executed repeatedly for the determined number of times. It updates the ranking of the chosen set. The decision is taken place based on the best ranked set. The evaluation pair is defined as <status s, action a>.

MDP followed by RL utilizes the learning Q classic function. It is a model free learning function which can start learning without any knowledge about the environment. It estimates the expected sum of reward whereas the optimal Q function finds the expected and achievable maximum sum of reward by following the particular policy on choosing the actions for specific states. Consider the equation (1).

$$Q(s, a) = \sum_{s'} E[R|s', \pi]P(s'|s, a) \tag{1}$$

where s is current state of the system, a is the action to be taken, π is the policy, s' is the expected state of the system when it is applied with the action, a. P is the probability of state changes, and E is the expected reward based on the learned policy.

The expected reward is defined as the reward of the current state (R|s) as well as the weighted (γ) reward of its subsequent state which is formulated in the equation (2).

$$E[R|s] = r + \gamma E[R + s'] \tag{2}$$

When the search space is large, MDP reduces the space and time with the support of this Q learning function. Fig. 2 depicts RL method.

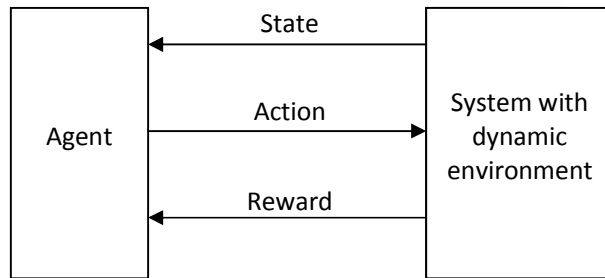


Fig. 2. Reinforcement learning method

Acquiring of correct data as training data as well as validation data are most important to get exact prediction results. When the observed data is huge in size, it would have noise and irrelevant data. Filtering of such unwanted data from the data set is most important one to get accurate results.

5 Conclusion

Thus the survey reports that the recent urge on the provision of self-managed, self-healing, and self-configured autonomous systems. It is somehow managed with the available AI-ML techniques. Review reveals the significance of various AI-ML techniques in setting right Smart Cloud. IBL-ML supervised learning techniques are suitable for fuzzy data set. RL techniques are for making the system to learn by its own experience and start its learning process in trial-and-error manner. It can deal with the system's dynamic environment by estimating the reward value with suitable policy and action. It attracts the researchers more to enhance their system as intelligent. RL stands itself as individual, and act as the companion of other heuristic as well as meta-heuristic algorithms.

6 Future Work

Even when the RL and other AI-ML algorithms are applied in various areas of research interest, still there is a demand for refinement in these algorithms to get optimal results. Refinement in RL method and finding of its applicable areas in Cloud Computing are the future works in plan.

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Competing Interests

Authors have declared that no competing interests exist.

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CSFC: A New Centroid Based Clustering Method to Improve the Efficiency of Storing and Accessing Small Files in Hadoop: Recent Advancement

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Abstract

In day to day life, the computer plays a major role, due to this advancement of technology collection of data from various fields are increasing. A large amount of data is produced by various fields because of IOT sensors for every second and is not easy to process. This large amount of data is called as Big data. A large number of small files also considered as Big data. It's not easy to process and store the small files in Hadoop. In the existing methods Merging technologies and Clustering Techniques are used to combine smaller files to large files up to 128 MB before sending it to HDFS in Hadoop. In the Proposed system CSFC (Clustering Small Files based on Centroid) Clustering Technique is used without mentioning the number of Clusters previously because if the clusters are mentioned before, all the files are clubbed within the limited number of clusters. In proposing system clusters are generated by depending on the number of related files in the dataset. The relevant files are combined up to 128 MB in a cluster. If any file is not relevant to the existing cluster or if the memory size reached 128MB then-new cluster will be generated and the file will be stored. It is easy to process the related files, comparing two relevant files. By using this method fetching data from the data node, it produces efficient result when comparing with other clustering techniques.

Keywords: Datanode; Hadoop distributed file system; Hadoop; Name node.

1 Introduction

Big data can be categorized as the huge volume of data that goes beyond traditional tools, systems or process handling capabilities. It is a requirement for an organization that manages such data to establish techniques and architectures to tackle this huge data with the generation of this big data at such a rapid pace.

1.1 Hadoop

Apache Hadoop is an open-source software platform that uses a single programming model to spread large data sets through commodity computer clusters. It is used to develop applications for data processing that are executed in a distributed computing environment. HADOOP-built applications operate on large data sets spread through commodity computer clusters.

1.2 HDFS framework

HDFS works efficiently with large files while comparing with small files. It contains two categories of nodes, one Name node and multiple data nodes, which is in the form of Master-slave architecture. The name node holds the metadata of each file which is going to store in the data node. Every block in data node can hold 128 MB of data. Small files are not up to 128MB in size. The metadata is stored in memory for quicker retrieval to minimize latency due to disk searches. Data Node is a daemon operating in the Hadoop Cluster on the 'Slave

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Node. Data frames are stored on the node of the slave. The data node performs read and write operations on disks.

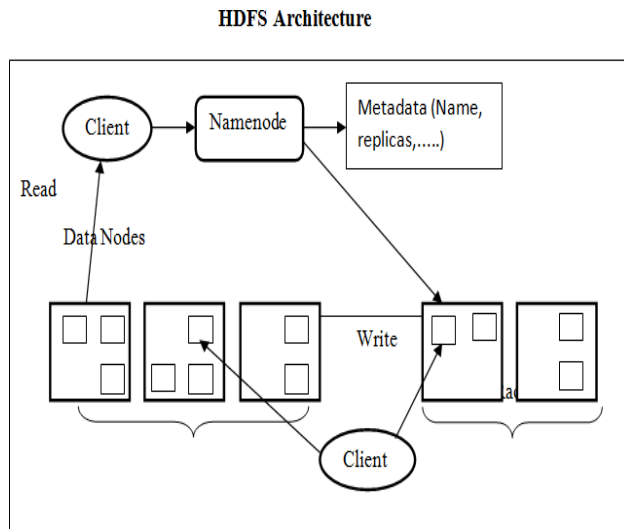


Fig. 1. HDFS architecture

1.3 Small files problem in Hadoop

A large number of small files are stored in Hadoop will occupy Name node memory separately for holding the metadata. The size of Name node is limited, suppose if 1 million of small files want to store means, then for each file 150 bytes of memory allocated separately in Name node and moreover single block in data node also allocated separately for each and every small file. Data node stores every file redundantly three times for security purpose. So accessing files from data node takes more time due to the large number of blocks occupied in data node.

If we want access a file first we have to refer HDFS name node for metadata [1,2]. In name node no difference of memory allocation for small files and big file, while accessing small file we need to access the name node frequently, because each and every small files are stored in separate data nodes. So it takes much time.

1.4 Clustering

Clustering is the method of splitting the entire data into groups based on some similarity of data. This is another famous clustering application. Multiple documents and you need to bring together related documents. Clustering allows us to organize these documents into the same clusters.

2 The Related Work

To solve small file problem in Hadoop, Small files are merged by using some techniques and then they are processed to produce good results in accessing the files [3]. Sequence File system. It is a binary file system provided by Hadoop. A sequence of binary key / value is composed of its data structure. The name of the file is called the key value and the result of the file is called the value, separated by the operation of the MapReduce block and independent storage. It merges a large amount of small files [4]. Hadoop Archive is a tool used to archive the small files in HDFS, Which reduces the utilization of Name node memory. Once HAR file is generated it cannot be modified, addition and deletion of files are not possible. Have to recreate the HAR file again [5]. Vorapongkitipun et al proposed two ways to improve HDFS and HAR. Two level indexing techniques are used to point the file. Two level indexing is not the easiest method.

Mohd Abdul Ahad et al. [6,7] proposed a method DM-sfs (Dynamic Merging based Small Files Storage) by categorizing all the files based on the size and they are combined based on the type of files. Merged files are stored in Name node and then they are processed. In that method the block is not utilized up to its level. The revised version of HAR called "NHAR" [8] to improve the use of metadata by memory and to provide space for files to be added to existing HAR which combines multiple small file into a large file. The resulting big archive consists of an index file and the small initial data. HAR's main drawback is that it does not allow file attachments.

Ahad, Mohd Abdul, and Biswas, Ranjit [8] in their proposed method they created methods to handle a large number of small files efficiently. First, they analyze the size of each file by using a file size analyzer algorithm and then the File type analyzer algorithm is used to combine files based on its type of dynamic merging strategy. Then the merged files are encrypted for security purpose, and then it's sent to HDFS. In this merging system files are arranged according to its size and then they are merged to overcome this step in our proposed system clustering technique is used to combine the related files using centroid.

Hooda, Hanu, Nandal, Rainu [9] in their proposed work k-means algorithm is to cluster the related files. In k-means algorithm cluster should be declared while starting the clustering technique after that centroid point is identified, then data sets are added which is close to the centroid. In that, all the files are clustered within that predefined number of clusters, but we are not sure that all the data sets are related to each other and they are fit within that predefined number of clusters to overcome this problem CSFC algorithm is generated.

3 The Proposed Work: Clustering Small Files Based on Centroid (CSFC)

The CSFC clustering technique: CSFC (Clustering Small files based on centroids): In the proposed system The huge number of small files is combined before sending it to the HDFS.

The combining technique is carried out by clustering CSFC based on the type of file and they are combined up to 128 MB of size because the block size is 128MB. If the file size is more than 128MB that will not create a problem because large files can be handled efficiently in Hadoop, but to keep the metadata efficiently the small files are combined up to 128 MB as large file.

3.1 Text file converter

This is responsible for identifying the file type. If the file type is text, then it is converted to numerical format of data for clustering the data are converted to numerical values to find the centroid value.

3.2 File cluster

By using the CSFC clustering algorithm First, it reads the data. The only one membership is generated because clusters are not predefined. Only one centroid is created for that single row Membership. The data are compared with centroid if the distance value is less than 0.5 then it is added to cluster of that centroid. Otherwise, data is considered as a new centroid and then new cluster will be generated. This is repeated for all the data. Before adding data into a cluster size will be checked if the cluster size less than 128 then that file will be added otherwise new cluster will be generated. In this algorithm number of clusters is not predefined because in CSFC clusters generated based on the related files by using the centroid.

3.3 Working on CSFC approach

It gets input from the client and then it is pre- processed to check whether the data is in numerical data or text file, If it is text file then it converted to numerical data, then the data is combined by using CSFC approach.

Converting text files to numerical data set:

- Input file
- Check the file type

- If file type is text then
- For every column in table do
- Assign initial value of=0
- For every row in column i do
- Calculate $Val=val+ data [i] [j]$
- Calculate average $=val/len(data)$
- Result=append(val)
- Forward the converted numerical dataset to CSFC Algorithm.

In the proposed algorithm It calculates the membership matrix for all the Filesets and then randomly center points are chosen. Based on the data difference ratio between the center and file, they are combined and the cluster will be generated. The number of clusters is not pre-defined in this algorithm. Depending upon the types of files and size of file's clusters are generated.

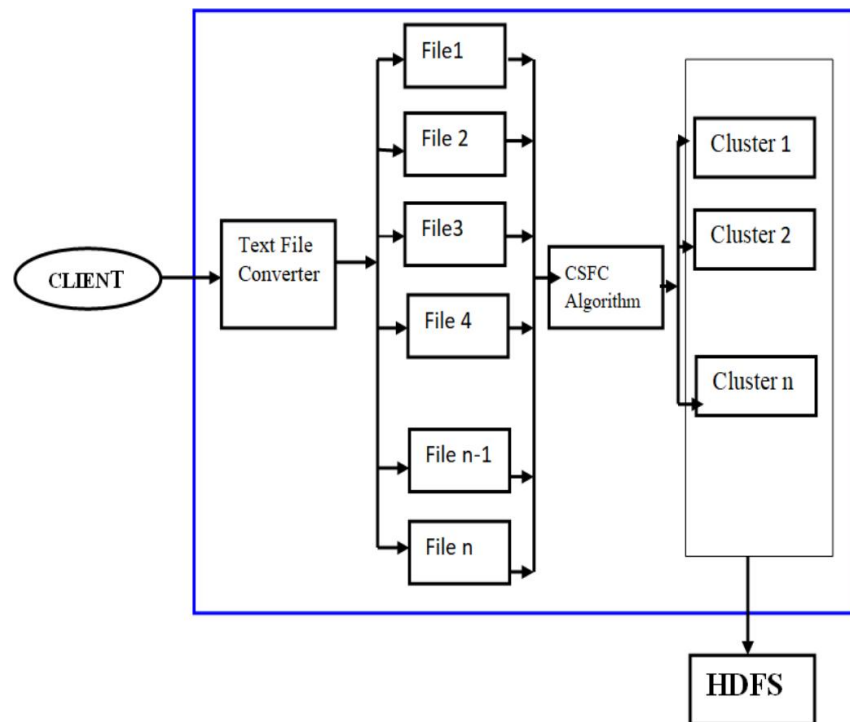


Fig. 2. Architecture of CSFC approach

CSFC Algorithm

- Read Input , Thresh hold, Iteration
- Generate one membership matrix for one row.
- Generate one centroid randomly
- Repeat
- Compare data with centroid
- If (distance (data and centroid < .5)
- If cluster size is less than 128 MB and Less than threshold
- Add files into clusters
- Else,
- generate new member ship matrix
- Assign data as new centroid and generate cluster
- Until iteration equal to zero

The generated clusters are sent to name node for storing the metadata. In data node it maintains details of files stored in each block of data node for easy access. In data node files are stored redundantly minimum three times in the rack for security purpose.

```
# Number of Clusters k = 0
# Maximum number of iterations MAX_ITER = 20
# Number of data points n = len(df)
# Fuzzy parameter m = 0.5
df_full = pd.read_csv("tinput.csv")
columns = list(df_full.columns)
features = columns[:len(columns)-1]
class_labels = list(df_full[columns[-1]])
df = df_full[features]
```

3.4 DDR comparison

Table 1. Data difference ratio between various algorithms

Algorithm	Average of DDR	Time taken in nanoseconds	No. of clusters
CSFC	2.328947368	2.5066853	9
k-means	2.394774775	4.9157381	2
Merging Algorithm	27.0201386	18.6666536	3

Data Difference ratio: It represents the data difference ratio values between each file and centroid value in the cluster. Data difference the average of a difference value between the centroid and data set.

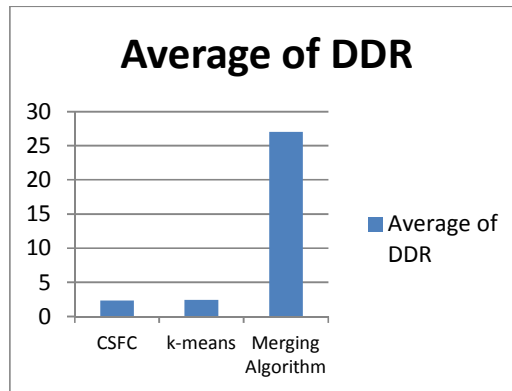


Fig. 3. Average of DDR value X-axis - algorithms, y- axis- values to show the average

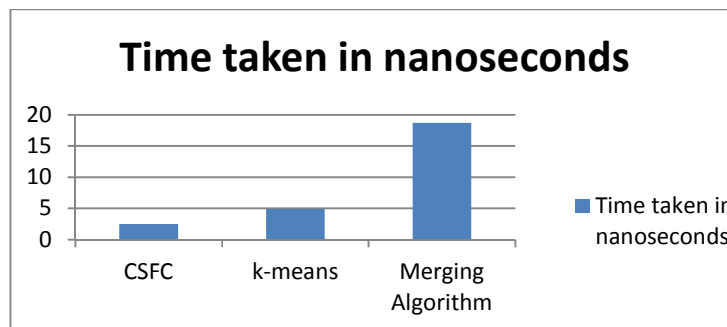


Fig. 4. X-axis time in nano seconds, Y-axis algorithms

Comparatively it is less in CSFC algorithm than K- means and merging algorithm. It represents how the files are related to each other.

If the DDR value decreases it increased the relationship between the data sets. Clustering time represents the time taken for generating clusters by various approaches in Nano seconds. If the time decreases then automatically processing speed increases. Our proposed CSFC is the efficient way compared with other algorithms. Totally 922 data sets are used with various file sizes from 1KB to 50MB. Number of files in the cluster will be different from algorithm to algorithm. The number of clusters generated in CFSC is high because the data difference value is decreased of clusters increased so all data in clusters are closely related files so its easy to retrieve comparing with other methods.

4 Conclusion

In advancement of technologies, data are generated in large number, but all the generated data are not large in size. For each and every second data are generated by IOT devices using sensors Clustering technique CSFC used to identify the file types they are clustered up to 128 MB in each cluster, otherwise new cluster will be generated. In this approach clusters are not predefined because of this the data are not clubbed in limited clusters. The related files are combined, if they are not related new cluster will be generated so it is easy to store and access the files in data node easily. The usage of Name node memory is reduced effectively by clustering. The execution time also reduced effectively with this approach when compared with existing work. In future enhancement data are encrypted and stored in Data node and retrieved efficiently.

Competing Interests

Authors have declared that no competing interests exist.

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Research on Tanimoto Coefficient Similarity Based Mean Shift Gentle Adaptive Boosted Clustering for Genomic Predictive Pattern Analytics

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Abstract

Gene expression data clustering is a significant problem to be resolved as it provides functional relationships of genes in a biological process. Finding co-expressed groups of genes is a challenging problem. To identify interesting patterns from the given gene expression data set, a Tanimoto Coefficient Similarity based Mean Shift Gentle Adaptive Boosted Clustering (TCS-MSGABC) Model is proposed. TCS-MSGABC model comprises two processes namely feature selection and clustering. In first process, Tanimoto Coefficient Similarity Measurement based Feature selection (TCSM-FS) is introduced to identify relevant gene features based on the similarity value for performing the genomic expression clustering. Tanimoto Coefficient Similarity Value ranges from '0' to '1', where '1' is highest similarity. The gene feature with higher similarity value is taken to perform clustering process. After feature selection, Mean Shift Gentle Adaptive Boosted Clustering (MSGABC) algorithm is carried out in TCS-MSGABC model to cluster the similar gene expression data based on the selected features. The MSGABC algorithm is a boosting method for combining the many weak clustering results into one strong learner. By this way, the similar gene expression data are clustered with higher accuracy with minimal time. Experimental evaluation of TCS-MSGABC model is carried out on factors such as clustering accuracy, clustering time and error rate with respect to number of gene data. The experimental results show that the TCS-MSGABC model is able to increase the clustering accuracy and also minimize clustering time of genomic predictive pattern analytics as compared to state-of-the-art works.

Keywords: Genomic; mean shift gentle adaptive boosted clustering; strong learner; tanimoto coefficient similarity; weak cluster; weight.

1 Introduction

A microarray database includes many microarray gene expression data. The irrelevant features present in the microarray database increase the time complexity of clustering algorithm. Higher dimensionality of the microarray database stimulates researchers to carry out feature selection using a variety of data mining techniques. The conventional feature selection and clustering algorithms faced some issues such as large number of feature genes, fewer numbers of samples and lack of proper validation as gene expression data is prone to outliers and noise. In order to overcome such limitations, TCS-MSGABC model is developed in this research work using Tanimoto Coefficient Similarity Measurement based Feature selection (TCSM-FS) and Mean Shift Gentle Adaptive Boosted Clustering (MSGABC) algorithms.

Subspace Weighting Co-Clustering (SWCC) was performed in [1] for high dimensional gene expression data. However, clustering performance was not enhanced. Gene ontology (GO) annotations based semi-supervised clustering algorithm called GO fuzzy relational clustering (GO-FRC) was designed in [2] for clustering of microarray gene expression data. But, GO-FRC consumes more time for efficiently grouping the gene data.

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A multivariate extension termed Relative Scan Statistics was carried out in [3] for comparison of two series in Bernoulli over frequent support. But, the clustering accuracy was not improved using Relative Scan Statistics. A random projection algorithm was introduced in [4] where random symmetric matrix was used to compute unsupervised clustering of dimensioned datasets like crystallographic structures. The clustering efficiency of PCA was not exact form of covariance/correlation matrix but it is symmetrical.

A new algorithm was introduced in [5] that measures similarity for individual gene groups and mixture of variants of hierarchical clustering to create the candidate groups. Semi-supervised consensus clustering (SSCC) was accomplished in [6] to enhance the robustness and quality of clustering results with a lower time complexity. However, error rate of gene expression data analysis was not reduced.

Rough-Fuzzy Clustering was carried out in [7] for grouping similar genes from microarray data with higher accuracy. But, clustering time taken for microarray data analysis was very higher. A novel gene selection method was developed in [8] depends on clustering where dissimilarity is estimated with help of kernel functions. However, clustering accuracy of gene expression data was poor.

Semi-supervised clustering was accomplished in [9] to resolve the gene expression data clustering problem with application of a multi-objective optimization. A feature selection based semi-supervised cluster ensemble framework (FS-SSCE) was employed in [10] for tumor clustering from bio-molecular data.

To addresses the above mentioned existing issues in genomic predictive pattern analytics, TCS-MSGABC model is introduced. The main contribution of TCS-MSGABC model is depicted in below,

- To minimize the time complexity of genomic data pattern clustering when compared to state-of-the-art works, Tanimoto Coefficient Similarity Measurement based Feature selection (TCSM-FS) algorithm is introduced in TCS-MSGABC model. On the contrary to conventional works, Tanimoto coefficient is a popular similarity coefficients used to calculate the similarity between pairs of the gene features. Besides to that, Tanimoto coefficient is an association coefficient which is used for binary data, that assigned a value that range from 1 (represents the complete similarity) and 0 (denotes no similarity).
- To enhance the accuracy of genomic data pattern clustering as compared to conventional works, Mean Shift Gentle Adaptive Boosted Clustering (MSGABC) algorithm is proposed in TCS-MSGABC model. On the contrary to state-of-the-art works, MSGABC is AdaBoost algorithm which increases overall performance of clustering by reducing both the training error and generalization error than other existing works. Hence, MSGABC algorithm attains higher clustering accuracy for genomic predictive pattern analysis.

The remaining structure of the paper is created as follows. In Section 2, TCS-MSGABC model is explained with the assist of the architecture diagram. In Section 3, Experimental settings are presented and the experimental result of TCS-MSGABC model is discussed in Section 4. Section5 depicts the literature survey. Section 6 depicts the conclusion of the paper.

2 Tanimoto Coefficient Similarity Based Mean Shift Gentle Adaptive Boosted Clustering Model

The Tanimoto Coefficient Similarity based Mean Shift Gentle Adaptive Boosted Clustering (TCS-MSGABC) model is introduced in order to increases the performance of genomic predictive pattern analytics through clustering with a lower time. The TCS-MSGABC model is proposed by combining Tanimoto Coefficient Similarity measurement and Gentle Adaptive Boost Clustering algorithm on the contrary to conventional works. In proposed TCS-MSGABC, Tanimoto Coefficient Similarity measurement is a feature selection technique which selects a subset of relevant features for building robust genomic predictive pattern analytics models. Through removing most irrelevant and redundant features from input gene expression dataset, Tanimoto Coefficient Similarity measurement improves feature selection performance and thereby finds the important features with higher accuracy.

After completing the feature selection process, TCS-MSGABC model utilizes the Gentle Adaptive Boost Clustering which is a variation of AdaBoost algorithm. The Gentle Adaptive Boost Clustering calculates one

weak hypothesis during the each iteration and finally unites these weak hypotheses in a linear manner. The Gentle Adaptive Boost Clustering designs the strong clustering results by means of optimizing the weighted least square error in each run. In addition to that, the Gentle Adaptive Boost Clustering increases weights for wrongly clustered instances exponentially. Thus, Gentle AdaBoost exactly clusters the similar gene pattern data together with higher accuracy and a minimal amount of time complexity. Through an efficient clustering of gene data, proposed TCS-MSGABC model gets better genomic predictive pattern analytics performance. The architecture diagram of the TCS-MSGABC model is demonstrated in Fig. 1.

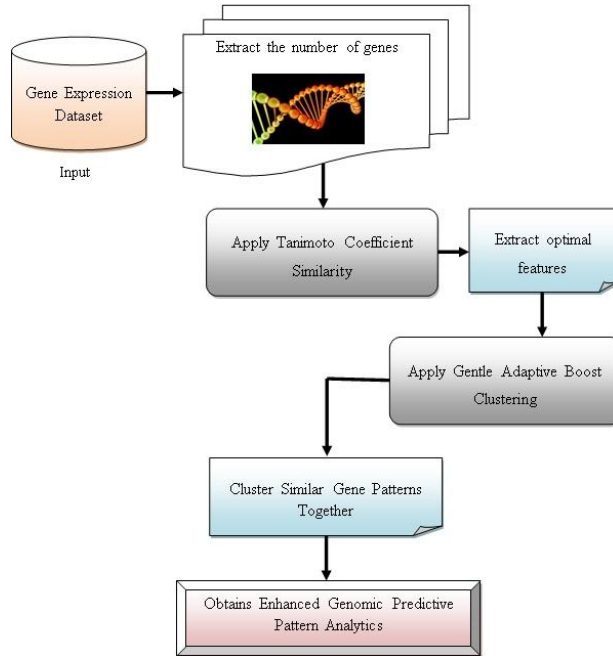


Fig. 1. Architecture diagram of TCS-MSGABC model

Fig. 1 presents the overall processes of the TCS-MSGABC model to obtain improved genomic predictive pattern analytics performance with minimum time complexity. As demonstrated in above Fig. 1, TCS-MSGABC model initially takes gene expression dataset (i.e. gene expression cancer RNA-Seq Dataset) as input which contains number of gene data and features. Subsequently, the TCS-MSGABC model applies Tanimoto Coefficient Similarity measurement with aim of discovering the most relevant features from the input gene expression dataset. Finally, TCS-MSGABC model applies Gentle Adaptive Boost Clustering with objective of grouping similar kind of gene pattern together. From that, TCS-MSGABC model significantly identifies interesting gene patterns from the given gene expression data set for efficient genomic predictive pattern analytics. The exhaustive process of TCS-MSGABC model is explained in the subsequent subsections.

3 Algorithms

3.1 Tanimoto coefficient similarity measurement based feature selection

Microarray gene expression data plays a significant role in feature selection as it supports for diagnosis and treatment of a variety of diseases. To reduce the time complexity of genomic predictive pattern analytics, a novel feature selection algorithm called Tanimoto Coefficient Similarity Measurement based Feature selection (TCSM-FS) is designed in TCS-MSGABC model. On the contrary to existing works, TCSM-FS algorithm is employed in proposed TCS-MSGABC model to find the significant gene features from an input Microarray gene expression dataset.

The TCSM-FS algorithm measures similarity between gene features. Based on measured similarity value, TCSM-FS algorithm remove irrelevant gene features that contain no useful information for the genomic pattern prediction and also remove redundant gene features that duplicate much or all of the information contained in one or more other features. From that, the TCSM-FS algorithm extracts the relevant gene features to reduce the time complexity of gene expression data analysis. Thus, TCSM-FS algorithm significantly avoids the curse of dimensionality for effective genomic predictive pattern analytics.

Let us assume an input microarray gene expression dataset contains number of gene features denoted as ' $\tau_1, \tau_2, \tau_3, \dots, \tau_M$ '. Here, ' M ' denotes total number of gene features in a given dataset. Followed by, the tanimoto coefficient similarity between the gene features is calculated mathematically as,

$$\emptyset(\tau_1, \tau_2) = \frac{M * \sum \tau_1 \tau_2}{\sum \tau_1^2 + \sum \tau_2^2 - \sum \tau_1 \tau_2} \quad (1)$$

From the above mathematical expression (1), ' $\emptyset(\tau_1, \tau_2)$ ' signifies a tanimoto similarity coefficient value, ' M ' point outs the number of gene features in microarray gene expression dataset, ' τ_1, τ_2 ' indicates a two gene features in dataset. Here, ' $\sum \tau_1^2$ ' designates a sum of squared score of the gene feature ' τ_1 ' and ' $\sum \tau_2^2$ ' signifies a sum of squared score of the gene feature ' τ_2 ' whereas ' $\sum \tau_1 \tau_2$ ' refers the sum of the product of the paired score of τ_1 and τ_2 . The tanimoto similarity coefficient value gives the output results from 0 to +1. In TCSM-FS algorithm, '+1' point outs the high similarity between the gene features and '0' denotes the low similarity between the gene features. Thus, TCSM-FS algorithm chooses gene features with high tanimoto similarity coefficient value for increasing the genomic predictive pattern analytics performance.

The algorithmic processes of TCSM-FS is explained as follows,

Input: Microarray Gene Expression Dataset, Number of gene features ' $\tau_1, \tau_2, \tau_3, \dots, \tau_M$ '
Output: Select relevant gene features
Begin
Step 1: For each input gene features
Step 2: Compute tanimoto similarity coefficient between the two gene features ' τ_1, τ_2 '
Step 3: If ' $\emptyset(\tau_1, \tau_2) = +1$ ' then
Step 4: Choose the features for genomic predictive pattern analytics
Step 5: Else
Step 6: Remove the gene features
Step 7: End if
Step 8: End For
End

Algorithm 1 Tanimoto Coefficient Similarity Measurement based Feature selection

Algorithm 1 depicts the step by step processes of TCSM-FS algorithm. As demonstrated in above algorithmic steps, at first TCSM-FS algorithm takes numbers of gene features from the microarray gene expression dataset as input. Subsequently the tanimoto similarity coefficient between the gene features are determined to identify the relevant and irrelevant gene features. If the tanimoto similarity coefficient value is '+1', then TCSM-FS algorithm extract gene features as more relevant for clustering genomic patterns. Otherwise, TCSM-FS algorithm eliminates gene features. With the chosen gene features, the similar types of genomic patterns are clustered which resulting in minimizing the time complexity.

3.2 Mean shift gentle adaptive boosted clustering

In TCS-MSGABC model, Mean Shift Gentle Adaptive Boosted Clustering (MSGABC) algorithm is a machine learning ensemble technique. The MSGABC algorithm changes the weak mean shift clustering results into the strong cluster for obtaining better genomic predictive pattern analytics performance. The MSGABC algorithm is a variant of AdaBoost classifier. In TCS-MSGABC model, Mean Shift clustering is taken as weak learner which does not give the higher clustering accuracy for genomic patterns analysis. Hence, MSGABC algorithm is proposed in TCS-MSGABC model through combining the number of weak clustering results into a strong learner. The constructed strong learner accurately groups the same types of genomic patterns together with a

minimal time complexity. The process involved in Mean Shift Gentle Adaptive Boosted Clustering is presented in below Fig. 2.

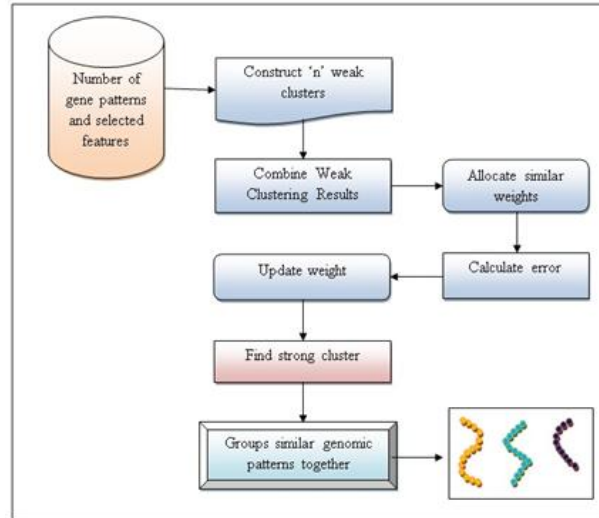


Fig. 2. Mean shift gentle adaptive boosted clustering for genomic patterns analysis

Fig. 2 presents the block diagram of MSGABC algorithm for increasing the clustering accuracy of genomic patterns. At first MSGABC algorithm designs a number of weak learner's i.e. Mean Shift Clustering for clustering the gene patterns. The weak learner is utilized in MSGABC algorithm is a centroid-based algorithm which works through grouping the each gene data to the mean of clusters in two-dimensional space. Thus, MSGABC algorithm removes irrelevant gene pattern data and thereby builds final set of clusters.

Let us consider a number of gene data in input microarray gene expression dataset is represented as ' $\rho_1, \rho_2, \rho_3, \dots, \rho_m$ ' where ' m ' denotes number of gene data in a given dataset. In Mean Shift Clustering, the mean ' ϑ ' is evaluated for all cluster in two dimensional spaces using below equation,

$$\vartheta = \frac{1}{m} \sum_{i=1}^m \rho_i \tag{2}$$

From the above mathematical expression (2), ' ϑ ' refers a mean of the cluster and ' ρ_i ' point outs the gene data. Here, the mean is determined based on the weighted average of the gene data in two-dimensional space. Subsequently, the nearby gene data are grouped into the corresponding mean using below mathematical representation,

$$GKF(\vartheta, \rho_i) = \exp\left(-\frac{\|\rho_i - \vartheta\|^2}{2v^2}\right) \tag{3}$$

From the above mathematical formulation (3), ' GKF ' denotes a Gaussian kernel function whereas ' $\|\rho_i - \vartheta\|^2$ ' represents a squared distance between the gene data and cluster mean in two dimensional space and ' v ' signifies a deviation from its mean. During the every iteration, each input gene data is grouped into a nearest cluster mean. The process of Mean Shift Clustering is continual until all the gene data are grouped into the clusters. The clustering accuracy of weak learner is not sufficient for accurate genomic patterns predictions. As a result, the outputs of all weak clusters are combined into a strong learner using below,

$$\beta = \sum_{i=1}^n w_i(\rho) \tag{4}$$

From the above formulation (4), ' β ' indicates a strong clustering results. Here, ' $w_i(\rho)$ ' denotes the output of the weak cluster.

Followed by, MSGABC algorithm initializes a similar weight for each weak cluster. After that, MSGABC algorithm evaluates the error rate by considering squared differentiation between the actual and estimated output of the each weak learner using below,

$$\delta = (\beta_o - w_i(\rho))^2 \quad (5)$$

From the above expression (5), ‘ δ ’ indicates a training error of the each weak cluster whereas ‘ β_o ’ signify the actual output of the weak cluster and ‘ $w_i(\rho)$ ’ denote the obtained output of the weak cluster. After finding the error, the weight of the all weak cluster is updated as follow,

$$\omega(t+1) = \omega_i(t) * e^{-\beta_i w_i(\rho)} \quad (6)$$

From the above mathematical representation (6), ‘ $\omega(t+1)$ ’ refers updated weight of the each weak cluster. Here, $\omega_i(t)$ denotes the initial weight of the weak cluster. Afterward the MSGABC algorithm identifies the weak cluster with minimum error as a strong learner. When the weak learner clusters the gene data incorrectly, the weight is increased. Otherwise the weight of the weak cluster is decreased in MSGABC algorithm. From that, the strong clustering results are obtained mathematically as,

$$\beta = arg \min \delta * w_i(\rho) \quad (7)$$

From the above formula (7), ‘ β ’ signifies a final output of the strong learner for clustering gene data with higher accuracy. Here, ‘ $arg \min \delta$ ’ represents a minimum error of the weak cluster. With the support of obtained strong learner, MSGABC algorithm correctly groups all the input gene data patterns into the different clusters with a minimal error rate. The algorithmic process of the MSGABC is described in below,

3.3 Mean shift gentle adaptive boosted clustering algorithm

Input: Number Of Gene Data ‘ $\rho_1, \rho_2, \rho_3, \dots, \rho_m$ ’ and Number of Gene Features ‘ $\tau_1, \tau_2, \tau_3, \dots, \tau_M$ ’

Output: Group similar gene data pattern together with higher accuracy

Begin

Step 1: For each gene data ‘ ρ_i ’

Step 2: Build ‘n’ weak clusters using (2) and (3)

Step 3: Combine all weak clusters results using (4)

Step 4: For each weak cluster ‘ $w_i(\rho)$ ’

Step 5: Define similar weights

Step 6: Calculate error ‘ δ ’ using (5)

Step 7: Update the weight ‘ $\omega(t+1)$ ’ using (6)

Step 8: Discover strong learner using (7)

Step 9: End for

Step 10: Get strong clustering results for genomic patterns analysis

Step 11: End for

End

Algorithm 2 Mean Shift Gentle Adaptive Boosted Clustering

Algorithm 2 presents the step by step process of MSGABC. At first, MSGABC algorithm creates a number of weak clusters for each input gene data. Subsequently, all the weak clustering results are aggregated. Then, the similar weight is given for all the weak clusters. Subsequently MSGABC algorithm computes training error for each weak clustering result. Consequently, the weight of each weak cluster is updated based on determined error value. Finally, the MSGABC algorithm finds the weak cluster with a minimal error rate as strong learner. This strong learner precisely clusters the similar gene data pattern together with a lower time complexity. By an effective clustering of gene data, TCS-MSGABC model obtains better genomic predictive pattern analytics performance with the minimum false positive rate as compared to conventional works.

4 Experimental Settings

In order to validate the proposed performance, TCS-MSGABC model is implemented in Java Language using gene expression cancer RNA-Seq Dataset [11] from UCI machine learning repository with 20531 attributes and 807 instances. This dataset contains random extraction of gene expressions of patient's i.e. different types of tumor: BRCA, KIRC, COAD, LUAD and PRAD. For conducting the experimental process, TCS-MSGABC model obtains various number of gene data in the range of 50-500 from gene expression cancer RNA-Seq Dataset. The efficiency of TCS-MSGABC model is evaluated in terms of clustering accuracy, clustering time and error rate with respect to diverse number of gene data.

The experimental evaluation of TCS-MSGABC model is conducted for several instances with respect to diverse number of gene data and averagely ten results are depicted in tabulation and graph. The experimental result of TCS-MSGABC model is compared against with two conventional works namely Gene ontology based fuzzy relational clustering (GO-FRC) [1] and Subspace Weighting Co-Clustering (SWCC) [2].

5 Results

In this section, the comparative result of TCS-MSGABC model is presented. The performance of TCS-MSGABC model is compared against with Gene ontology based fuzzy relational clustering (GO-FRC) [1] and Subspace Weighting Co-Clustering (SWCC) [2] respectively. The effectiveness of TCS-MSGABC model is measured along with the following metrics with the help of tables and graphs.

5.1 Clustering accuracy

Clustering accuracy 'CA' calculates the ratio of number of gene data correctly grouped to the total number of gene data taken for experimental process. The clustering accuracy is measured as follows,

$$CA = \frac{y_{ac}}{m} * 100 \quad (8)$$

From the above mathematical equation (8), 'y_{ac}' signifies number of gene data accurately clustered and 'm' indicates a total number of gene data. The clustering accuracy is determined in terms of percentage (%).

Sample Mathematical Calculation:

- **Existing GO-FRC:** Number of gene data precisely clustered is 31 and the total number of gene data is 50. Then the clustering accuracy is obtained as,

$$CA = \frac{31}{50} * 100 = 62 \%$$

- **Existing SWCC:** Number of gene data properly clustered is 36 and the total number of gene data is 50. Then the clustering accuracy is acquired as,

$$CA = \frac{36}{50} * 100 = 72 \%$$

- **Proposed TCS-MSGABC:** Number of gene data perfectly clustered is 45 and the total number of gene data is 50. Then the clustering accuracy is determined as,

$$CA = \frac{45}{50} * 100 = 90 \%$$

The tabulation result analysis of clustering accuracy for genomic pattern predictive analytics is depicted in below Table 1. When carried outing the experimental process using 400 gene data, the proposed TCS-MSGABC model achieves 93% clustering accuracy whereas existing GO-FRC [1] and SWCC [2] obtains 78% and 81%

respectively. Hence, it is considerable that the clustering accuracy of gene data patterns using proposed TCS-MSGABC model is higher when compared to other works [1] and [2].

Table 1. Tabulation result of clustering accuracy

Number of gene data (m)	Clustering Accuracy (%)		
	<i>GO-FRC</i>	<i>SWCC</i>	<i>TCS-MSGABC</i>
50	62	72	90
100	66	76	92
150	73	81	94
200	79	80	93
250	81	86	94
300	81	85	92
350	80	83	91
400	78	81	93
450	76	81	90
500	77	80	89

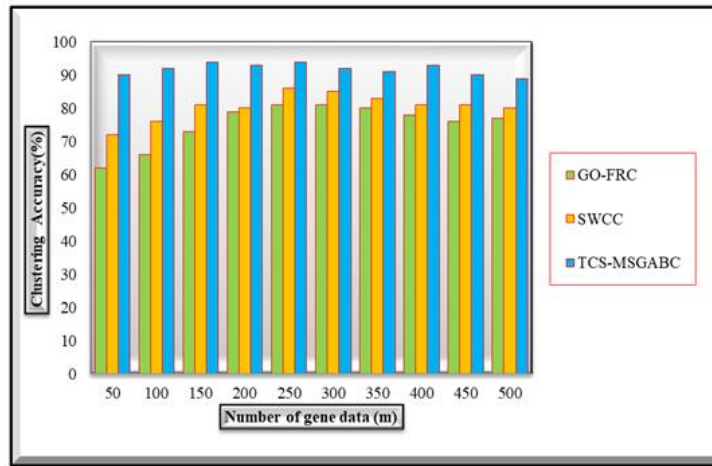


Fig. 3. Experimental result of clustering accuracy versus number of gene data

Fig. 3 presents the impact of clustering accuracy for genomic pattern predictive analytics with respect to various number of gene data using three methods namely GO-FRC [1] and SWCC [2] and proposed TCS-MSGABC model. As exposed in above graphical figure, proposed TCS-MSGABC model presents enhanced accuracy to group the similar genomic patterns together when compared with GO-FRC [1] and SWCC [2]. This is because of processes of TCSM-FS and MSGABC algorithms in proposed TCS-MSGABC model on the contrary to conventional works.

TCSM-FS algorithm discovers the relevant and irrelevant gene features. After completing the feature selection process, TCS-MSGABC model utilizes MSGABC algorithm where it makes a strong learner with a minimal training error for efficiently cluster the gene data patterns based on selected features. Thus, proposed TCS-MSGABC model improves the ratio of number of gene data correctly grouped as compared to conventional works. Therefore, proposed TCS-MSGABC model obtains enhanced clustering accuracy for analyzing genomic patterns by 23 % as compared to GO-FRC [1] and 14 % as compared to SWCC [2] respectively.

5.2 Clustering time

Clustering time calculates the time needed for clustering similar genomic data together. The clustering time is estimated as,

$$CT = m * t(csg) \tag{9}$$

From the above formulation (9), ‘ $t(csg)$ ’ indicates a time utilized for grouping single gene data and ‘ m ’ signifies a total number of gene data. The clustering time is computed in terms of milliseconds (ms).

Sample mathematical calculation:

- **Existing GO-FRC:** time employed to cluster single gene data is 0.48 ms and the total number of gene data is 50. Then the clustering time is evaluated as,

$$CT = 50 * 0.48 = 24 \text{ ms}$$

- **Existing SWCC:** the time consumed to cluster single gene data is 0.41 ms and the total number of gene data is 50. Then the clustering time is estimated as,

$$CT = 50 * 0.41 = 21 \text{ ms}$$

- **Proposed TCS-MSGABC:** time required to cluster single gene data is 0.36 ms and the total number of gene data is 50. Then the clustering time is computed as,

$$CT = 50 * 0.36 = 18 \text{ ms}$$

The experimental result analysis of clustering time involved during process of genomic pattern predictive analytics is shown in below Table 2. When conducting the experimental evaluation by taking 300 gene data, the proposed TCS-MSGABC model gets 72 ms clustering time whereas conventional GO-FRC [1] and SWCC [2] attains 93 ms and 87 ms respectively. Therefore, it is significant that the clustering time of gene data patterns using proposed TCS-MSGABC model is lower as compared to other works [1] and [2].

Table 2. Tabulation result of clustering time

Number of gene data (n)	Clustering Time (ms)		
	<i>GO-FRC</i>	<i>SWCC</i>	<i>TCS-MSGABC</i>
50	24	21	18
100	40	33	31
150	59	48	44
200	74	60	54
250	90	75	65
300	93	87	72
350	105	91	84
400	116	108	92
450	126	117	104
500	130	125	115

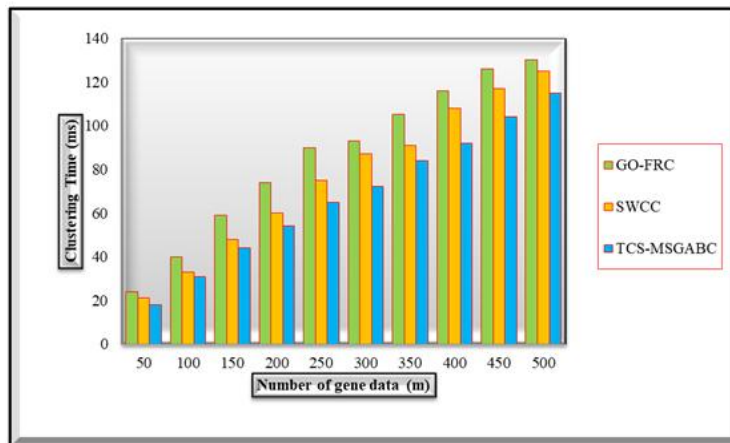


Fig. 4. Experimental result of clustering time versus number of gene data

Fig. 4 depicts the performance result of clustering time for genomic pattern predictive analytics based on different number of gene data using three methods namely GO-FRC [1] and SWCC [2] and proposed TCS-MSGABC model. As shown in above graphical diagram, proposed TCS-MSGABC model takes minimal amount of clustering time to group the related genomic patterns together when compared to GO-FRC [1] and SWCC [2]. This is owing to processes of TCSM-FS and MSGABC algorithms in proposed TCS-MSGABC model on the contrary to state-of-the-art works.

With the application of TCSM-FS algorithmic process, proposed TCS-MSGABC model take outs the significant gene features to decrease the time complexity of gene expression data clustering. From that, TCS-MSGABC model considerably minimizes the curse of dimensionality for effectual genomic predictive pattern analytics. Besides to that, proposed TCS-MSGABC model changes the weak mean shift clustering results into the strong cluster in order to correctly group the gene data with minimal time consumption. Accordingly, proposed TCS-MSGABC model minimizes the time required for clustering similar genomic data together as compared to conventional works. As a result, proposed TCS-MSGABC model attains reduced clustering time of genomic patterns analysis by 22% as compared to GO-FRC [1] and 11% as compared to SWCC [2] respectively.

5.3 Error rate

Error Rate ‘ER’ computes the ratio of number of gene data wrongly grouped to the total number of gene data. The error rate is measured as,

$$ER = \frac{y_{ic}}{m} * 100 \tag{10}$$

From the above expression (10), ‘ y_{ic} ’ refers a number of gene data incorrectly clustered and ‘ m ’ denotes a total number of gene data in given dataset. The error rate is estimated in terms of percentage (%).

Sample Mathematical Calculation:

- **Existing GO-FRC:** number of gene data mistakenly clustered is 19 and the total number of gene data is 50. Then the error rate is obtained as,

$$ER = \frac{19}{50} * 100 = 38 \%$$

- **Existing SWCC:** number of gene data poorly clustered is 14 and the total number of gene data is 50. Then the error rate is calculated as,

$$ER = \frac{14}{50} * 100 = 28 \%$$

- **Proposed TCS-MSGABC:** number of gene data inaccurately clustered is 5 and the total number of gene data is 50. Then the error rate is measured as,

$$ER = \frac{5}{50} * 100 = 10 \%$$

The performance result analysis of error rate involved during clustering process of gene data is demonstrated in below Table 3. When accomplishing the experimental work with 450 gene data, the proposed TCS-MSGABC model obtains 10% error rate whereas state-of-the-art works GO-FRC [1] and SWCC [2] gains 24% and 19% respectively. As a result, it is expressive that the error rate of gene data patterns clustering using proposed TCS-MSGABC model is minimal when compared to other works [1] and [2].

Fig. 5 depicts comparative result of error rate for clustering genomic data patterns along with varied number of gene data using three methods namely GO-FRC [1] and SWCC [2] and proposed TCS-MSGABC model. As illustrated in above graphical illustration, proposed TCS-MSGABC model gives lower error rate to perfectly

cluster the interrelated genomic patterns together when compared to GO-FRC [1] and SWCC [2]. This is due to processes of MSGABC algorithm in proposed TCS-MSGABC model on the contrary to state-of-the-art works.

Table 3. Tabulation result of error rate

Number of gene data (n)	Error Rate (%)		
	<i>GO-FRC</i>	<i>SWCC</i>	<i>TCS-MSGABC</i>
50	38	28	10
100	34	24	8
150	27	19	6
200	22	21	8
250	19	14	6
300	19	15	8
350	20	17	9
400	22	19	7
450	24	19	10
500	23	20	11

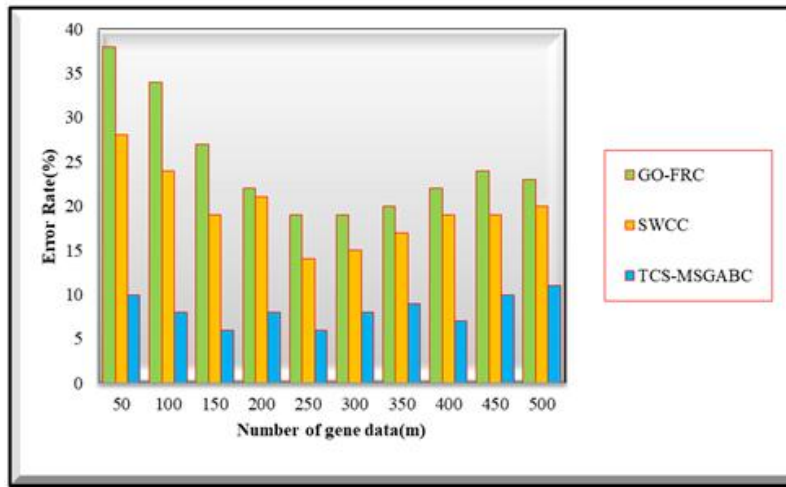


Fig. 5. Experimental result of error rate versus number of gene data

By using the concepts of MSGABC algorithm, proposed TCS-MSGABC model get enhanced clustering performance for gene patterns analysis by considering both the training error and generalization error as compared to state-of-the-art works. As a result, proposed TCS-MSGABC model achieves better strong clustering result for grouping the genomic patterns. Hence, proposed TCS-MSGABC model reduces the ratio of number of gene data wrongly grouped as compared to conventional works. Consequently, proposed TCS-MSGABC model gets minimized error rate of gene data clustering by 65% as compared to GO-FRC [1] and 57% as compared to SWCC [2] respectively.

6 Literature Survey

Gradual shadowed set was utilized in [12] for grouping similar gene expression with a lower error rate. A Multi objective Variable Length PSO-Based Approach was introduced in [13] for discovering non-redundant gene markers from microarray data and reducing time complexity.

Spectral ensemble biclustering (SEB) was employed in [14] for enhancing efficiency and scalability of gene expression data. Noise Resistant Generalized Parametric Validity Index of Clustering was presented in [15] for gene expression data analysis.

A novel method was designed in [16] for grouping of short time-course gene expression data with dissimilar replicates. Mutual Information-Based Supervised Attribute Clustering was performed in [17] for determining biologically considerable gene clusters with excellent predictive capability.

Projective clustering ensemble (PCE) was employed in [18] to get better quality of clustering gene expression data through dimensionality reduction. A review of different techniques designed for analysis of microarray data was presented in [19].

Tight clustering algorithm was employed in [20] to minimize time complexity of large microarray gene expression data. An evolutionary uncertain data-clustering algorithm was designed in [21] to determine the similarities among sets of gene expression clusters.

7 Conclusion

An effective TCS-MSGABC model is designed with aim of enhancing the performance of genomic pattern predictive analytics via performing clustering with higher accuracy and minimal time. The aim of TCS-MSGABC model is obtained with the support of TCSM-FS and MSGABC algorithmic process on the contrary to conventional works. The proposed TCS-MSGABC model attains enhanced ratio of number of gene data that are accurately clustered by designing a strong learner with a lower error as compared to state-of-the-art works. Moreover, proposed TCS-MSGABC model gets minimal amount of time complexity to efficiently group the genomic data patterns as compared to existing works. Thus, proposed TCS-MSGABC model gives better accuracy, time and error rate for genomic pattern predictive analysis performance as compared to state-of-the-art works. The efficiency of TCS-MSGABC model is evaluated in terms of clustering accuracy, clustering time, and error rate and compared with state of the art works. The experimental result demonstrates that TCS-MSGABC model provides better performance with an enhancement of clustering accuracy and minimization of clustering time to find similar gene patterns when compared to state-of-the-art works.

Competing Interests

Authors have declared that no competing interests exist.

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Mathematical Modeling on a Typical Three Species Ecology

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Abstract

In this chapter, we discuss the stability analysis of mathematical modeling on a typical three species ecology. The system comprises of a commensal (S_1), two hosts S_2 and S_3 i.e., S_2 and S_3 both benefit S_1 , without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1 , S_2 . Here all three species are having limited resources quantized by the respective carrying capacities. The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths N_1 , N_2 , N_3 of S_1 , S_2 , S_3 respectively. In all, eight equilibrium points of the model are identified. The system would be stable, if all the characteristic roots are negative, in case they are real and have negative real parts, in case they are complex. Further, the trajectories of the perturbations over the equilibrium points are illustrated.

Keywords: Commensal; equilibrium point; host; stable; trajectories; unstable.

1 Introduction

Mathematical Modeling plays a vital role in providing insight into the mutual relationships (positive, negative) between the interacting species. The general concepts of Modeling in Biological Science have been discussed by several authors Ma [1], Murray [2] and Sze-Bi Hsu [3]. Srinivas [4] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad and et al [6-11] investigated continuous and discrete models on two, three and four species syn-ecosystems.

Ecology is the study of the interactions between organisms and their environment. The organisms include animals and plants, the environment includes the surroundings of animals. So ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution-commensalism, prey-predator, competition and so on. Significant researches in the area of theoretical ecology have been discussed by Gillman [12] and by Kot [13]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Arumugam [14] and Sharma [15].

Fig. 1 shows a schematic diagram of the interaction under study. Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other, the helping one (S_2) is called the host species. A common example is an animal using a tree for shelter-tree (host) does not get any benefit from the animal (commensal).

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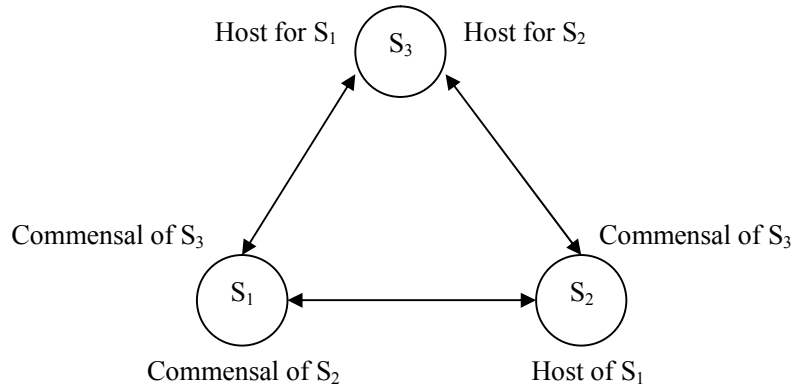


Fig. 1. Schematic sketch of the three species ecology

2 Basic Equations of the Model

The model equations for a typical three species ecology is given by the following system of first order non-linear ordinary differential equations employing the following notation.

Notation Adopted:

- $N_i(t)$: The population strength of S_i at time t , $i = 1, 2, 3$.
 t : Time instant.
 a_i : Natural growth rate of S_i , $i = 1, 2, 3$.
 a_{ii} : Self inhibition coefficients of S_i , $i = 1, 2, 3$.
 a_{12}, a_{13} : Interaction coefficients of S_1 due to S_2 and S_1 due to S_3 .
 a_{23} : Interaction coefficient of S_2 due to S_3 .
 $k_i = \frac{a_i}{a_{ii}}$: Carrying capacities of S_i , $i = 1, 2, 3$.

Further the variables N_1, N_2, N_3 are non-negative and the model parameters $a_1, a_2, a_3, a_{11}, a_{22}, a_{33}, a_{13}, a_{23}$ are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad (1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{23} N_2 N_3 \quad (2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 \quad (3)$$

3 Equilibrium States

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0, i = 1, 2, 3 \quad (4)$$

(i) Fully washed out state.

$$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(ii) States in which two of the tree species are washed out and third is not.

$$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3; E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_3, \bar{N}_3 = 0$$

$$E_4 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$$

(iii) Only one of the three species is washed out while the other two are not.

$$E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}}, \bar{N}_3 = k_3; E_6 : \bar{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = k_3$$

$$E_7 : \bar{N}_1 = k_1 + \frac{a_{12}k_2}{a_{11}}, \bar{N}_2 = k_2, \bar{N}_3 = 0$$

iv) The co-existent state or normal steady state.

$$E_8 : \bar{N}_1 = k_1 + \frac{a_{12}}{a_{11}} \left(k_2 + \frac{a_{23}k_3}{a_{22}} \right) + \frac{a_{13}k_3}{a_{11}}, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{11}}, \bar{N}_3 = k_3$$

4 Stability of the Equilibrium States

Let $N = (N_1, N_2, N_3) = \bar{N} + U$

Where $U = (u_1, u_2, u_3)^T$ is a small perturbation over the state $\bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3)$.

The basic equations (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as

$$\frac{dU}{dt} = AU \quad (5)$$

with

$$A = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 + a_{13}\bar{N}_3 & a_{12}\bar{N}_1 & a_{13}\bar{N}_1 \\ 0 & a_2 - 2a_{22}\bar{N}_2 + a_{23}\bar{N}_3 & a_{23}\bar{N}_2 \\ 0 & 0 & a_3 - 2a_{33}\bar{N}_3 \end{bmatrix} \quad (6)$$

The characteristic equation for the system is

$$\det [A - \lambda I] = 0 \quad (7)$$

The equilibrium state is stable, if all the roots of the equation (7) are negative, in case they are real or have negative real parts, in case they are complex.

4.1 Fully washed out state

To discuss the stability of equilibrium point $E_1(0,0,0)$. Let us consider small deviations $u_1(t)$, $u_2(t)$, $u_3(t)$ from the steady state.

$$\text{i.e., } N_i(t) = \bar{N}_i + u_i(t), \quad i = 1, 2, 3$$

where $u_i(t)$ is a small perturbations in the species S_i .

The corresponding linearized equations for the perturbations u_1 , u_2 , u_3 are

$$\frac{du_i}{dt} = a_i u_i, \quad i = 1, 2, 3 \tag{8}$$

The characteristic equation is given by

$$\Pi(\lambda - a_i) = 0, \quad i = 1, 2, 3 \tag{9}$$

The roots a_1 , a_2 , a_3 of which are all positive. Hence the fully washed out state is **unstable** and the solutions of the equations (8) are

$$u_i = u_{i0} e^{a_i t}, \quad i = 1, 2, 3 \tag{10}$$

where u_{10} , u_{20} , u_{30} are the initial values of u_1 , u_2 , u_3 respectively.

Trajectories of Perturbations: The trajectories in $u_1 - u_2$, $u_2 - u_3$, $u_1 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1}, \left(\frac{u_2}{u_{20}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_2}, \left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{a_1} \text{ respectively.}$$

4.2 Equilibrium point $E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$

The corresponding linearized equations for the perturbations are

$$\frac{du_1}{dt} = \alpha_1 u_1, \quad \frac{du_2}{dt} = \alpha_2 u_2, \quad \frac{du_3}{dt} = -a_3 u_3 \tag{11}$$

$$\text{where } \alpha_1 = a_1 + a_{13} k_3 > 0 \text{ and } \alpha_2 = a_2 + a_{23} k_3 \tag{12}$$

The characteristic equation is $(\lambda - \alpha_1)(\lambda - \alpha_2)(\lambda + a_3) = 0$ and its characteristic roots are α_1 , α_2 and $-a_3$. Since only one root is negative, hence the state is **unstable**. The equations (11) yield the solutions.

$$u_1 = u_{10} e^{\alpha_1 t}, \quad u_2 = u_{20} e^{\alpha_2 t}, \quad u_3 = u_{30} e^{-a_3 t} \tag{13}$$

The trajectories in $u_1 - u_2$, $u_2 - u_3$, $u_1 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{\alpha_2} = \left(\frac{u_2}{u_{20}}\right)^{\alpha_1}, \left(\frac{u_2}{u_{20}}\right)^{-\alpha_3} = \left(\frac{u_3}{u_{30}}\right)^{\alpha_2}, \left(\frac{u_1}{u_{10}}\right)^{-\alpha_3} = \left(\frac{u_3}{u_{30}}\right)^{\alpha_1} \text{ respectively.}$$

4.3 Equilibrium point $E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0$.

The linearized equations are

$$\frac{du_1}{dt} = \beta_1 u_1, \frac{du_2}{dt} = -a_2 u_2 + a_{23} k_3 u_3, \frac{du_3}{dt} = a_3 u_3 \quad (14)$$

$$\text{where } \beta_1 = a_1 + a_{12} k_2 > 0 \quad (15)$$

$(\lambda - \beta_1)(\lambda + a_2)(\lambda - a_3) = 0$ is the characteristic equation of (14) and its characteristic roots are $\beta_1, -a_2, a_3$. Since two of these three roots are positive, hence the state is **unstable** and the solutions are given by

$$u_1 = u_{10} e^{\beta_1 t}, u_2 = (u_{10} - \alpha_3) e^{-a_2 t} + \alpha_3 e^{a_3 t}, u_3 = u_{30} e^{a_3 t} \quad (16)$$

$$\text{where } \alpha_3 = \frac{a_{23} k_3 u_{30}}{a_2 + a_3} \quad (17)$$

$$\frac{u_2}{u_{20}} = A \left(\frac{u_1}{u_{10}}\right)^{\frac{-a_2}{\beta_1}} + \frac{\alpha_3}{u_{20}} \left(\frac{u_1}{u_{10}}\right)^{\frac{a_3}{\beta_1}}, \left(\frac{u_1}{u_{10}}\right)^{a_3} = \left(\frac{u_3}{u_{30}}\right)^{\beta_1}; \frac{u_2}{u_{20}} = A \left(\frac{u_3}{u_{30}}\right)^{\frac{-a_2}{a_3}} + \frac{\alpha_3}{u_{20}} \left(\frac{u_3}{u_{30}}\right)$$

are the trajectories.

$$\text{where } A = \frac{u_{10} - \alpha_3}{u_{20}}$$

4.4 Equilibrium point $E_4 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this point the linearized equations for the perturbations u_1, u_2, u_3 are

$$\frac{du_1}{dt} = -a_1 u_1 + a_{12} k_1 u_2 + a_{13} k_1 u_3, \frac{du_2}{dt} = a_2 u_2, \frac{du_3}{dt} = a_3 u_3 \quad (18)$$

The characteristic equation is $(\lambda + a_1)(\lambda - a_2)(\lambda - a_3) = 0$.

The characteristic roots are $-a_1, a_2, a_3$. Since only one root is negative, hence the state is **unstable**. The equations (18) yield the solutions.

$$u_1 = (u_{10} - \alpha - \beta) e^{-a_1 t} + \alpha e^{a_2 t} + \beta e^{a_3 t}, u_2 = u_{20} e^{a_2 t}, u_3 = u_{30} e^{a_3 t} \quad (19)$$

$$\text{where } \alpha = \frac{a_{12} k_1 u_{20}}{a_1 + a_2} > 0 \text{ and } \beta = \frac{a_{13} k_1 u_{30}}{a_1 + a_3} > 0 \quad (20)$$

The trajectories of perturbations are given by

$$\frac{u_1}{u_{10}} = B \left(\frac{u_2}{u_{20}} \right)^{\frac{-a_1}{a_2}} + \frac{\alpha}{u_{10}} \left(\frac{u_2}{u_{20}} \right) + \frac{\beta}{u_{10}} \left(\frac{u_2}{u_{20}} \right)^{\frac{a_3}{a_2}}, \left(\frac{u_2}{u_{20}} \right)^{a_3} = \left(\frac{u_3}{u_{30}} \right)^{a_2}$$

$$\frac{u_1}{u_{10}} = B \left(\frac{u_3}{u_{30}} \right)^{\frac{-a_1}{a_3}} + \frac{\alpha}{u_{10}} \left(\frac{u_3}{u_{30}} \right)^{\frac{a_2}{a_3}} + \frac{\beta}{u_{10}} \left(\frac{u_3}{u_{30}} \right).$$

$$\text{where } B = \frac{u_{10} - \alpha - \beta}{u_{10}}$$

4.5 Equilibrium point $E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}}, \bar{N}_3 = k_3$

The corresponding linearized equations for this equilibrium point are

$$\frac{du_1}{dt} = \gamma_1 u_1, \frac{du_2}{dt} = -\gamma_2 u_2 + \gamma_3 u_3, \frac{du_3}{dt} = -a_3 u_3 \quad (21)$$

where

$$\left. \begin{aligned} \gamma_1 &= \left(a_1 + a_{12}k_2 + \frac{a_{12}a_{23}k_3}{a_{22}} + a_{13}k_3 \right) > 0, \gamma_2 = (a_2 + a_{23}k_3) > 0 \\ \gamma_3 &= \left(a_{23}k_2 + \frac{a_{23}^2k_3}{a_{22}} \right) > 0 \end{aligned} \right\} \quad (22)$$

$(\lambda - \gamma_1)(\lambda + \gamma_2)(\lambda + a_3) = 0$ is the characteristic equation and the characteristic roots are $\gamma_1, -\gamma_2, -a_3$. Since one of these three roots is positive, hence the state is **unstable**. The solutions of (21) are given by

$$u_1 = u_{10} e^{\gamma_1 t}, u_2 = (u_{20} - \gamma) e^{-\gamma_2 t} + \gamma e^{-a_3 t}, u_3 = u_{30} e^{-a_3 t} \quad (23)$$

$$\text{where } \gamma = \frac{\gamma_3 u_{30}}{\gamma_2 - a_3} \quad (24)$$

The trajectories in $u_1 - u_2, u_1 - u_3, u_2 - u_3$ planes are

$$\frac{u_2}{u_{20}} = (1 - C) \left(\frac{u_1}{u_{10}} \right)^{\frac{-\gamma_2}{\gamma_1}} + C \left(\frac{u_1}{u_{10}} \right)^{\frac{-a_3}{\gamma_1}}, \left(\frac{u_1}{u_{10}} \right)^{-a_3} = \left(\frac{u_3}{u_{30}} \right)^{\gamma_1},$$

$$\frac{u_2}{u_{20}} = (1 - C) \left(\frac{u_3}{u_{30}} \right)^{\frac{\gamma_2}{a_3}} + C \left(\frac{u_3}{u_{30}} \right) \text{ respectively.}$$

$$\text{where } C = \frac{\gamma_3 u_{30}}{(\gamma_2 - a_3) u_{20}}$$

4.6 Equilibrium point $E_6 : \bar{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = k_3$

The corresponding linearized equations for the perturbations u_1, u_2, u_3 are

$$\frac{du_1}{dt} = -\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3, \frac{du_2}{dt} = \beta_2 u_2, \frac{du_3}{dt} = -a_3 u_3 \quad (25)$$

$$\text{where } \delta_1 = a_1 + a_{13}k_3 > 0, \delta_2 = a_{12}k_1 + \frac{a_{12}a_{13}k_3}{a_{11}} > 0 \quad (26)$$

$$\delta_3 = a_{13}k_1 + \frac{a_{13}^2 k_3}{a_{11}} > 0, \beta_2 = a_2 + a_{23}k_3 > 0 \quad (27)$$

The characteristic equation for which is $(\lambda + \delta_1)(\lambda - \beta_2)(\lambda + a_3) = 0$.

The characteristic roots are $-\delta_1, \beta_2, -a_3$. Since one root is positive, hence the state is **unstable**. The equations yield the solutions.

$$u_1 = (u_{10} - \rho_1 - \rho_2)e^{-\delta_1 t} + \rho_1 e^{\beta_2 t} + \rho_2 e^{-a_3 t}, u_2 = u_{20} e^{\beta_2 t}, u_3 = u_{30} e^{-a_3 t} \quad (28)$$

$$\text{where } \rho_1 = \frac{\delta_2 u_{20}}{\delta_1 + \beta_2}, \rho_2 = \frac{\delta_3 u_{30}}{\delta_1 - a_3} \quad (29)$$

The trajectories in $u_1 - u_2, u_2 - u_3, u_1 - u_3$ planes are

$$\frac{u_1}{u_{10}} = D \left(\frac{u_2}{u_{20}} \right)^{\frac{-\delta_1}{\beta_2}} + \frac{\rho_1}{u_{10}} \left(\frac{u_2}{u_{20}} \right) + \frac{\rho_2}{u_{10}} \left(\frac{u_2}{u_{20}} \right)^{\frac{-a_3}{\beta_2}}, \left(\frac{u_2}{u_{20}} \right)^{-a_3} = \left(\frac{u_3}{u_{30}} \right)^{\beta_2},$$

$$\frac{u_1}{u_{10}} = D \left(\frac{u_3}{u_{30}} \right)^{\frac{\delta_1}{a_3}} + \frac{\rho_1}{u_{10}} \left(\frac{u_3}{u_{30}} \right)^{\frac{-\beta_2}{a_3}} + \frac{\rho_2}{u_{10}} \left(\frac{u_3}{u_{30}} \right) \text{ respectively.}$$

$$\text{where } D = \frac{u_{10} - \rho_1 - \rho_2}{u_{10}}$$

4.7 Equilibrium point $E_7 : \bar{N}_1 = k_1 + \frac{a_{12}k_2}{a_{11}}, \bar{N}_2 = k_2, \bar{N}_3 = 0$

In this point the corresponding linearized equations are

$$\frac{du_1}{dt} = -\mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3, \frac{du_2}{dt} = -a_2 u_2 + a_{23}k_3 u_3, \frac{du_3}{dt} = a_3 u_3 \quad (30)$$

where

$$\mu_1 = a_1 + a_{12}k_2 > 0, \mu_2 = a_{12}k_1 + \frac{a_{12}^2 k_2}{a_{11}} > 0, \mu_3 = a_{13}k_1 + \frac{a_{12}a_{13}k_2}{a_{11}} > 0 \quad (31)$$

The characteristic equation for (30) is $(\lambda + \mu_1)(\lambda + a_2)(\lambda - a_3) = 0$

The characteristic roots are $-\mu_1, -a_2, a_3$. Since one root is positive, hence the state is **unstable**.

The equations yield the solutions.

$$u_1 = (u_{10} - \delta - \mu) e^{-\mu t} + \delta e^{-a_2 t} + \mu e^{a_3 t}, u_2 = (u_{20} - \rho) e^{-a_2 t} + \rho e^{a_3 t}, u_3 = u_{30} e^{a_3 t} \quad (32)$$

$$\text{where } \delta = \frac{\mu_2(u_{30} - \delta)}{\mu_1 - a_2}, \mu = \frac{\rho\mu_2 + \mu_3 u_{30}}{a_3 + \mu_1}, \rho = \frac{a_{23} k_3 u_{30}}{a_2 + a_3} \quad (33)$$

The trajectories in u_1 - u_3 , u_2 - u_3 planes are

$$\frac{u_1}{u_{10}} = D \left(\frac{u_3}{u_{30}} \right)^{\frac{-\mu_1}{a_3}} + \delta \left(\frac{u_3}{u_{30}} \right)^{\frac{-a_2}{a_3}} + \mu \left(\frac{u_3}{u_{30}} \right), \frac{u_2}{u_{20}} = \left(\frac{u_{20} - \delta}{u_{20}} \right) \left(\frac{u_2}{u_{30}} \right)^{\frac{-a_2}{a_3}} + \rho \left(\frac{u_3}{u_{30}} \right) \text{ respectively.}$$

$$\text{where } D = \frac{u_{10} - \delta - \mu}{u_{10}}$$

4.8 Normal steady state

The corresponding linearized equations for the normal steady state are

$$\frac{du_1}{dt} = -\sigma_1 u_1 + a_{12} \eta_1 u_2 + a_{12} \eta_1 u_3, \frac{du_2}{dt} = -\sigma_2 u_2 + \sigma_3 u_3, \frac{du_3}{dt} = -a_3 u_3 \quad (34)$$

where

$$\left. \begin{aligned} \sigma_1 &= \left(a_1 + a_{12} k_2 + \frac{a_{12} a_{23} k_3}{a_{22}} + a_{13} k_3 \right) > 0 \\ \sigma_2 &= (a_2 + a_{23} k_3) > 0, \sigma_3 = \left(a_{23} k_3 + \frac{a_{23}^2 k_3}{a_{22}} \right) > 0 \end{aligned} \right\} \quad (35)$$

$(\lambda + \sigma_1)(\lambda + \sigma_2)(\lambda + a_3) = 0$ is the characteristic equation and its roots are $-\sigma_1, -\sigma_2, -a_3$. Since all the three roots are negative, hence the normal steady state is **stable**. The equations yield the solutions.

$$u_1 = [u_{10} - (\varphi_2 + \varphi_3)] e^{-\sigma_1 t} + \varphi_2 e^{-\sigma_2 t} + \varphi_3 e^{-a_3 t}, u_2 = (u_{20} - \varphi_1) e^{-\sigma_2 t} + \varphi_1 e^{-a_3 t}, u_3 = u_{30} e^{-a_3 t} \quad (36)$$

$$\text{where } \varphi_1 = \frac{\sigma_3 u_{30}}{\sigma_2 - a_3}, \varphi_2 = \frac{a_{12} \eta_1 (\varphi_1 - u_{20})}{\sigma_2 - \sigma_1}, \varphi_3 = \frac{a_{12} \eta_1 \varphi_1 + a_{13} \eta_1 u_{30}}{\sigma_3 - \sigma_1} \quad (37)$$

It can be noticed that $u_1 \rightarrow 0, u_2 \rightarrow 0$ and $u_3 \rightarrow 0$ as $t \rightarrow \infty$

$\frac{u_1}{u_{10}} = E \left(\frac{u_3}{u_{30}} \right)^{\frac{\sigma_1}{a_3}} + \phi_2 \left(\frac{u_3}{u_{30}} \right)^{\frac{\sigma_2}{a_3}} + \phi_3 \left(\frac{u_3}{u_{30}} \right)$, $\frac{u_2}{u_{20}} = F \left(\frac{u_3}{u_{30}} \right)^{\frac{\sigma_2}{a_3}} + \phi_1 \left(\frac{u_3}{u_{30}} \right)$ are the trajectories in $u_1 - u_3$, $u_2 - u_3$ planes respectively.

where $E = \frac{u_{10} - (\phi_2 + \phi_3)}{u_{10}}$, $F = \frac{u_{20} - \phi_1}{u_{20}}$.

5 Conclusion

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between three species (S_1, S_2, S_3) with the population relations. This chapter deals with an investigation on a typical three species ecology. The system comprises of a commensal (S_1), two hosts S_2 and S_3 i.e., S_2 and S_3 both benefit S_1 , without getting themselves effected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1, S_2 . It is observed that, in all eight equilibrium states, only the normal steady state is stable and rest of them are unstable.

Competing Interests

Author has declared that no competing interests exist.

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Record Values in the Estimation of a Parameter of Some Distributions with Known Coefficient of Variation

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Abstract

We discuss the general technique of estimating the location parameter of certain distributions with known coefficient of variation by record values. Also we estimate the location parameters of normal distribution, logistic distribution and exponential distribution using upper record values. Also we include a real life data to estimate the location parameter of a logistic distribution using upper record values.

Keywords: Normal distribution; logistic distribution; exponential distribution; best linear unbiased estimation; coefficient of variation; record values.

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1 Introduction

Let X_1, X_2, \dots be a sequence of independent observations arising from a population. An observation X_j will be called an upper record value (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i, \forall i < j$. The first observation X_1 is taken as the initial record R_1 . The next record R_2 is the observation following R_1 which is greater than R_1 and so on. The records R_1, R_2, \dots as defined above are sometimes referred to as the sequence of upper records. Similarly, an observation X_j will be called a lower record value if its value is less than that of all previous observations.

Record values and associated statistics are of great importance in several real life problems involving weather, economic and sports data. The statistical study of record values started with Chandler [1] and has now spread in different directions. Resnick [2] and Shorrock [3] documented the asymptotic theory of records. Glick [4] provides a survey of the literature on records. For a detailed discussion on the developments in the theory and applications of record values, see Arnold et al. [5], Nagaraja [6], Nevzorov and Balakrishnan [7] and Ahsanullah [8].

Let X_1, X_2, \dots be a sequence of independent observations arising from a population with absolutely continuous cumulative distribution function (cdf) $F_X(x)$ and pdf $f_X(x)$. If we write R_n to denote the n^{th} upper record value, then its pdf is given by (see Arnold et al. [5], p.10),

$$f_{R_n}(x) = \frac{1}{(n-1)!} [-\log\{1 - F_X(x)\}]^{n-1} f_X(x). \quad (1)$$

The joint pdf of the m^{th} and n^{th} upper record values R_m and R_n for $m < n$ is given by (see Arnold et al. [5], p.11),

$$f_{R_m, n}(x_1, x_2) = \frac{1}{(m-1)!(n-m-1)!} [-\log\{1 - F_X(x_1)\}]^{m-1} \frac{f_X(x_1)}{1 - F_X(x_1)}$$

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$$\times [-\log\{1 - F_X(x_2)\} + \log\{1 - F_X(x_1)\}]^{n-m-1} f_X(x_2), x_1 < x_2. \quad (2)$$

If we write L_n to denote the n^{th} lower record value then its pdf is given by,

$$f_{L_n}(x) = \frac{1}{(n-1)!} [-\log\{F_X(x)\}]^{n-1} f_X(x). \quad (3)$$

The joint pdf of the m^{th} and n^{th} lower record values L_m and L_n for $m < n$ is given by,

$$f_{L_{m,n}}(x_1, x_2) = \frac{1}{(m-1)!(n-m-1)!} [-\log\{F_X(x_1)\}]^{m-1} \frac{f_X(x_1)}{F_X(x_1)} \\ \times [-\log\{F_X(x_2)\} + \log\{F_X(x_1)\}]^{n-m-1} f_X(x_2), x_2 < x_1. \quad (4)$$

The study of record values in many ways parallels the study of order statistics, indeed they are inextricably related. In any physical phenomena, occurrence of extreme situations attracts the attention of the experimenter. Record value theory developed in statistics, enter forcibly into the study of such situations. In these situations record values and record breaking data are also found useful to draw inferences about the underlying population distribution. The problems associated with statistical inference based on record values are discussed by several authors. For a detailed discussion on the developments in the theory and applications of record values, see Arnold et al. [5], Nevzorov [9], Nagaraja [6], Nevzorov and Balakrishnan [7] and Ahsanullah [8]. Hence the main objective of this chapter is to estimate the location parameter of some distributions with known coefficient of variation using record values. Most of the results established in this chapter are available in Sajeevkumar and Irshad [10] and Sajeevkumar[11].

In section 2, we consider the general technique of estimating the location parameter of a distribution with known coefficient of variation by upper record values.

A continuous random variable X is said to have the normal distribution with location parameter μ and scale parameter $c\mu$, if its pdf is given by,

$$f(x; \mu, c\mu) = \frac{1}{c\mu\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2c^2\mu^2}\right\}, \mu > 0, c > 0, x \in R. \quad (5)$$

The above mentioned normal model with known coefficient of variation is useful in environmental studies where μ represents mean concentration level of a particular chemical or pollutant (in air or water), and the standard deviation $c\mu$ is directly proportional to mean concentration level with the proportionality constant c being known from either past studies or physical characteristics of the environmental setup (see Guo and Pal [12]). For other applications of this distribution see also Gleser and Healy [13], Khan[14], Khan[15], Kunte[16], Sinha [17], Hinkley [18], Joshi and Sathe [19], Gupta et. al. [20] and Sen [21]. Estimating the mean of normal distribution with known coefficient of variation by order statistics are discussed by Thomas and Sajeevkumar[22] and Sajeevkumar[23]. Estimation of the mean of truncated normal distribution with known coefficient of variation are discussed by Khatri and Ratani [24]. Recently testing the mean of a normal distribution with known coefficient of variation are elucidated by Bhat and Rao [25]. Hence in section 3 we consider the problem of estimation of the mean of a normal distribution with known coefficient of variation by record values.

The logistic distribution is a well-known and widely used statistical distribution because of its simplicity and its historical importance as a growth curve (see Erkelens [26]). It has several important applications in biological, actuarial, industrial and engineering fields. Some applications of order statistics from the logistic distribution in the fields of life-testing and reliability studies have been mentioned by Lawless [27] and Mann et al. [28]. Also logistic distribution has a shape similar to that of normal distribution, which makes it simpler and also profitable on suitable

occasions to replace the normal by the logistic distribution to simplify the analysis without too great discrepancies in the respective theories. Hence in section 4, we consider the problem of estimation of the mean of logistic distribution with known coefficient of variation by record values.

A continuous random variable X is said to have the logistic distribution with location parameter μ and scale parameter $c\mu$, if its pdf is given by (see Sajeevkumar and Thomas [29] and Sajeevkumar[23]),

$$f(x; \mu, c\mu) = \frac{\pi}{\sqrt{3}} \frac{\exp\left\{\frac{-\pi}{\sqrt{3}}\left(\frac{x-\mu}{c\mu}\right)\right\}}{c\mu \left[1 + \exp\left\{\frac{-\pi}{\sqrt{3}}\left(\frac{x-\mu}{c\mu}\right)\right\}\right]^2}, \quad \mu > 0, c > 0, x \in R. \quad (6)$$

In section 5, we consider the problem of estimation of the location parameter μ of the exponential distribution with known coefficient of variation by record values. The compact form of the derived estimators are also elucidated in this section. A continuous random variable X is said to follow the exponential distribution with location parameter μ and scale parameter $a\mu$, if its pdf is given by, (See, Sajeevkumar[11])

$$f(x; \mu, a\mu) = \begin{cases} \frac{1}{a\mu} \exp\left\{-\frac{x-\mu}{a\mu}\right\}, & x \geq \mu, \mu > 0, a > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

In the last section of this chapter, we consider a real life data, to illustrate the theory developed in section 2.

2 Estimation of the Location Parameter of a Distribution when the Scale Parameter is Proportional to the Location Parameter by Upper Record Values

In this section we consider the family \mathcal{G} of all absolutely continuous distributions which depend on a location parameter μ and a scale parameter σ , such that $\sigma = c\mu$. Throughout this chapter we assume that c is known. Then any distribution belongs to \mathcal{G} has a pdf of the form,

$$f(x; \mu, c\mu) = \frac{1}{c\mu} f_0\left(\frac{x-\mu}{c\mu}\right), \quad \mu > 0, c > 0, x \in R. \quad (8)$$

If in the pdf defined in (8), μ is the mean and $c\mu$ is the standard deviation, then c is the known coefficient of variation. Let $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ be the first n upper record values arising from (8) and let $Y_{U(1)}, Y_{U(2)}, \dots, Y_{U(n)}$ be the corresponding upper record values arising from the standard form of the pdf defined in (8). Let us now denote $E(Y_{U(n)})$ by α_n , $E(Y_{U(n)}^2)$ by α_n^2 , $Var(Y_{U(n)})$ by $\beta_{n,n}$, $E(Y_{U(m)}Y_{U(n)})$ by $\alpha_{m,n}$ and $Cov(Y_{U(m)}, Y_{U(n)})$ by $\beta_{m,n}$. Let $\mathbf{X}_{(U)} = (X_{U(1)}, X_{U(2)}, \dots, X_{U(n)})'$ be the vector of first n upper record values arising from (8) and let $\mathbf{Y}_{(U)} = (Y_{U(1)}, Y_{U(2)}, \dots, Y_{U(n)})'$ be the corresponding vector of first n upper record values arising from the standard form of (8).

Now we derive the BLUE of μ based on upper record values and is given in the following theorem.

Theorem 2.1. *Suppose $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ are the first n upper record values arising from (8). Let $Y_{U(1)}, Y_{U(2)}, \dots, Y_{U(n)}$ be the corresponding upper record values arising from the standard form of the distribution defined in (8). Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)'$, $\mathbf{B} = ((\beta_{i,j}))$, $1 \leq i \leq j \leq n$, be the vector of means and dispersion matrix respectively of $\mathbf{Y}_{(U)} = (Y_{U(1)}, Y_{U(2)}, \dots, Y_{U(n)})'$. Then the BLUE $\tilde{\mu}_1$ of the parameter μ is given by,*

$$\tilde{\mu}_1 = \frac{(c\boldsymbol{\alpha} + \mathbf{1})'\mathbf{B}^{-1}}{(c\boldsymbol{\alpha} + \mathbf{1})'\mathbf{B}^{-1}(c\boldsymbol{\alpha} + \mathbf{1})} \mathbf{X}_{(u)} \quad (9)$$

and

$$\text{Var}(\tilde{\mu}_1) = \frac{c^2 \mu^2}{(c\boldsymbol{\alpha} + \mathbf{1})'\mathbf{B}^{-1}(c\boldsymbol{\alpha} + \mathbf{1})}, \quad (10)$$

where $\mathbf{X}_{(U)} = (X_{U(1)}, X_{U(2)}, \dots, X_{U(n)})'$ and $\mathbf{1}$ is a column vector of n ones.

Proof. Given $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ are first n upper record values arising from (8).

$$\text{Let } E(Y_{U(i)}) = \alpha_i, \quad i = 1, 2, \dots, n,$$

$$\text{Var}(Y_{U(i)}) = \beta_{i,i}, \quad i = 1, 2, \dots, n$$

and

$$\text{Cov}(Y_{U(i)}, Y_{U(j)}) = \beta_{i,j}, \quad 1 \leq i \leq j \leq n.$$

Then we have,

$$\frac{X_{U(i)} - \mu}{c\mu} \stackrel{d}{=} Y_{U(i)}, \quad i = 1, 2, \dots, n.$$

Therefore,

$$E(X_{U(i)}) = (c\alpha_i + 1)\mu, \quad i = 1, 2, \dots, n \quad (11)$$

$$\text{Var}(X_{U(i)}) = c^2 \mu^2 \beta_{i,i} \quad (12)$$

and

$$\text{Cov}(X_{U(i)}, X_{U(j)}) = c^2 \mu^2 \beta_{i,j}. \quad (13)$$

Using (11) to (13) one can also write,

$$E(\mathbf{X}_{(U)}) = (c\boldsymbol{\alpha} + \mathbf{1})\mu \quad (14)$$

and

$$D(\mathbf{X}_{(U)}) = \mathbf{B}c^2\mu^2, \quad (15)$$

where $\mathbf{1}$ is a column vector of n ones, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)'$, $\mathbf{B} = ((\beta_{i,j}))$ $1 \leq i \leq j \leq n$ and $\mathbf{X}_{(U)} = (X_{U(1)}, X_{U(2)}, \dots, X_{U(n)})'$. Then by generalized Gauss-Markov setup, the BLUE $\tilde{\mu}_1$ of the parameter μ is given by,

$$\tilde{\mu}_1 = \frac{(c\boldsymbol{\alpha} + \mathbf{1})'\mathbf{B}^{-1}}{(c\boldsymbol{\alpha} + \mathbf{1})'\mathbf{B}^{-1}(c\boldsymbol{\alpha} + \mathbf{1})} \mathbf{X}_{(U)}$$

and

$$\text{Var}(\tilde{\mu}_1) = \frac{c^2 \mu^2}{(c\boldsymbol{\alpha} + \mathbf{1})'\mathbf{B}^{-1}(c\boldsymbol{\alpha} + \mathbf{1})}.$$

Thus the theorem is proved.

Clearly $\tilde{\mu}_1$ can be written as a linear function of $X_{U(i)}$ as,

$$\tilde{\mu}_1 = \sum_{i=1}^n b_i X_{U(i)}, \text{ where } b_i, i = 1, 2, \dots, n \text{ are constants.}$$

3 Estimation of the Mean of the Normal Distribution with Known Coefficient of Variation by Record Values

A continuous random variable X is said to have the normal distribution with location parameter μ and scale parameter $c\mu$, if its pdf is given by,

$$f(x; \mu, c\mu) = \frac{1}{c\mu\sqrt{2\pi}} \exp \left\{ -\frac{(x - \mu)^2}{2c^2\mu^2} \right\}, \quad \mu > 0, c > 0, x \in R. \quad (16)$$

We will write $N(\mu, c\mu)$ to denote the normal distribution defined in (16). The mean and variance of the above distribution are given by $E(X) = \mu$ and $Var(X) = c^2\mu^2$, where c is the known coefficient of variation. The record values and associated inference arising from the two parameter normal distribution $N(\mu, \sigma)$ are discussed by Balakrishnan and Chan [30]. Using the results of Balakrishnan and Chan [30] and also using the results based on upper record values given section 2, we have evaluated the coefficients of $X_{U(i)}$ in the BLUE $\tilde{\mu}_1$, $Var(\tilde{\mu}_1)$ defined in (10) for the parameter μ involved in (16), for $c = 0.15(0.05)0.30$ and $n = 2(1)10$ and are given in Table 1.

Table 1. Coefficients of $X_{U(i)}$ in the BLUE $\tilde{\mu}_1$, $V_1 = \frac{Var(\tilde{\mu}_1)}{\mu^2}$, for different values of c .

n	c	Coefficients										V_1	
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}		
2	0.15	0.14458	0.75336										0.01335
3		0.11204	0.07408	0.65628									0.01021
4		0.09422	0.06206	0.04537	0.60197								0.00853
5		0.08245	0.05462	0.03957	0.03162	0.56539							0.00745
6		0.07414	0.04895	0.03582	0.02814	0.02363	0.53845						0.00668
7		0.06773	0.04476	0.03259	0.02599	0.02111	0.01903	0.51713					0.00610
8		0.06265	0.04147	0.03028	0.02387	0.01937	0.01787	0.01406	0.50070				0.00565
9		0.05858	0.03857	0.02819	0.02260	0.01822	0.01628	0.01330	0.01276	0.48582			0.00526
10		0.05505	0.03631	0.02672	0.02065	0.01691	0.01681	0.01187	0.01198	0.00954	0.47414		0.00494
2		0.2	0.09996	0.76233									0.02216
3	0.07521		0.05916	0.65775								0.01635	
4	0.06192		0.04839	0.03627	0.59825							0.01332	
5	0.05322		0.04188	0.03103	0.02517	0.55786						0.01142	
6	0.04720		0.03696	0.02773	0.02202	0.01873	0.52793					0.01008	
7	0.04258		0.03338	0.02489	0.02013	0.01643	0.01506	0.50421				0.00909	
8	0.03897		0.03061	0.02290	0.01826	0.01487	0.01404	0.01085	0.48570			0.00830	
9	0.03612		0.02816	0.02110	0.01718	0.01389	0.01259	0.01020	0.01002	0.46909		0.00768	
10	0.03365		0.02629	0.01988	0.01544	0.01273	0.01319	0.00890	0.00932	0.00720	0.45589	0.00715	

Table 1. Continued

n	c	Coefficients										V_1
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
2	0.25	0.06058	0.76638									0.03231
3		0.44660	0.04654	0.65432								0.02304
4		0.03620	0.03726	0.02882	0.58874							0.01836
5		0.03066	0.03180	0.02424	0.02000	0.54483						0.01547
6		0.02691	0.02768	0.02144	0.01723	0.01488	0.51228					0.01348
7		0.02403	0.02474	0.01901	0.01564	0.01283	0.01199	0.48657				0.01201
8		0.02180	0.02250	0.01735	0.01402	0.01148	0.01112	0.00842	0.46642			0.01088
9		0.13502	0.07250	0.05105	0.03992	0.03206	0.02796	0.02317	0.02136	0.48424		0.00997
10		0.01858	0.01901	0.01486	0.01162	0.01040	0.00883	0.00761	0.00734	0.00551	0.43414	0.00923
2		0.3	0.02600	0.76635								
3	0.01941		0.03590	0.64494								0.02999
4	0.01575		0.02820	0.02274	0.57554							0.02341
5	0.01324		0.02380	0.01881	0.01587	0.52865						0.01944
6	0.01158		0.02048	0.01651	0.01347	0.01184	0.49403					0.01674
7	0.01027		0.01814	0.01448	0.01217	0.01004	0.00960	0.46683				0.01478
8	0.00927		0.01639	0.01315	0.01079	0.00889	0.00887	0.00656	0.44550			0.01328
9	0.00853		0.01480	0.01191	0.01011	0.00822	0.00773	0.00613	0.00637	0.42669		0.00977
10	0.00784		0.01366	0.01114	0.00878	0.00807	0.00686	0.00594	0.00584	0.00422	0.41161	0.01111

4 Estimation of the Mean of the Logistic Distribution with Known Coefficient of Variation by Record Values

A continuous random variable X is said to have the logistic distribution with location parameter μ and scale parameter $c\mu$, if its pdf is given by,

$$f(x; \mu, c\mu) = \frac{\pi}{\sqrt{3}} \frac{\exp\left\{\frac{-\pi}{\sqrt{3}}\left(\frac{x-\mu}{c\mu}\right)\right\}}{c\mu \left[1 + \exp\left\{\frac{-\pi}{\sqrt{3}}\left(\frac{x-\mu}{c\mu}\right)\right\}\right]^2}, \quad \mu > 0, c > 0, x \in R. \quad (17)$$

We will write $LD(\mu, c\mu)$ to denote the logistic distribution defined in (17). The mean and variance of the above distribution are given by $E(X) = \mu$ and $Var(X) = c^2\mu^2$, where c is the known coefficient of variation. In this section we consider only upper record values and hence whenever we use the term record values in this section it means upper record values. Applications of record values arising from two parameter logistic distribution are elucidated by Balakrishnan et al. [31]. We have evaluated the means, variances and covariances of the record values arising from the standard form of the distribution defined in (17). Using these values and also using the results based on record values given in section 2, we have evaluated the coefficients of $X_{U(i)}$ in the BLUE $\tilde{\mu}_1$, $Var(\tilde{\mu}_1)$ defined in (10) for the parameter μ involved in (17), for $c = 0.15(0.05)0.30$ and $n = 2(1)10$ and are given in Table 2.

Table 2. Coefficients of $X_{U(i)}$ in the BLUE $\tilde{\mu}_1$, $V_1 = \frac{Var(\tilde{\mu}_1)}{\mu^2}$, for different values of c .

n	c	Coefficients										V_1	
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}		
2	0.15	0.12665	0.76877										0.01569
3		0.19037	0.24273	0.43213									0.01416
4		0.19387	0.23233	0.12722	0.29060								0.01352
5		0.19339	0.22418	0.12498	0.06517	0.22058							0.01316
6		0.19162	0.21917	0.12268	0.06409	0.03147	0.18632						0.01292
7		0.18949	0.21540	0.12079	0.06311	0.03101	0.01533	0.16874					0.01272
8		0.18723	0.21224	0.11913	0.06221	0.03057	0.01513	0.00754	0.15916				0.01254
9		0.18496	0.20941	0.11756	0.06139	0.03019	0.01491	0.00746	0.00376	0.15348			0.01238
10		0.18273	0.20673	0.11609	0.06060	0.02983	0.01472	0.00734	0.00372	0.00192	0.01497		0.01222
2		0.2	0.05671	0.79847									
3	0.12921		0.23777	0.44895									0.02298
4	0.13472		0.22443	0.11421	0.31403								0.02163
5	0.13492		0.21392	0.11125	0.05964	0.24600							0.02084
6	0.13333		0.20713	0.10824	0.05816	0.02892	0.21186						0.02027
7	0.13117		0.20188	0.10574	0.05682	0.02827	0.01415	0.19352					0.01980
8	0.12884		0.19744	0.10353	0.05560	0.02767	0.01388	0.00698	0.18281				0.01938
9	0.12648		0.19335	0.10146	0.05448	0.02714	0.01357	0.00686	0.00350	0.17585			0.01900
10	0.12418		0.18971	0.09954	0.05342	0.02664	0.01331	0.00671	0.00345	0.00179	0.17079		0.01864

Table 2. Continued

n	c	Coefficients										V ₁
		a ₁	a ₂	a ₃	a ₄	a ₅	a ₆	a ₇	a ₈	a ₉	a ₁₀	
2	0.25	-0.00421	0.81861									0.03772
3		0.07713	0.23147	0.45886								0.03274
4		0.08533	0.21548	0.10238	0.32939							0.03037
5		0.08680	0.20291	0.09888	0.05445	0.26279						0.02895
6		0.08593	0.19457	0.09534	0.05265	0.02649	0.22845					0.02791
7		0.08428	0.18806	0.09240	0.05103	0.02569	0.01301	0.20914				0.02704
8		0.08236	0.18254	0.08981	0.04956	0.02496	0.01267	0.00642	0.19718			0.02628
9		0.08039	0.17761	0.08741	0.04823	0.02432	0.01229	0.00627	0.00324	0.18889		0.02558
10		0.07846	0.17304	0.08519	0.04698	0.02372	0.01198	0.00609	0.00317	0.00166	0.18252	0.02493
2		0.3	-0.05671	0.83071								
3	0.03324		0.22432	0.46327								0.04295
4	0.04457		0.20604	0.09177	0.33842							0.03931
5	0.04771		0.19176	0.08791	0.04969	0.27291						0.03708
6	0.04795		0.18216	0.08403	0.04765	0.02424	0.23822					0.03542
7	0.04716		0.17466	0.08081	0.04583	0.02333	0.01119	0.21793				0.03406
8	0.04598		0.16832	0.07799	0.04419	0.02251	0.01155	0.00590	0.20478			0.03287
9	0.04470		0.16270	0.07541	0.04272	0.02179	0.01113	0.00573	0.00298	0.19525		0.03179
10	0.04341		0.15754	0.07305	0.04136	0.02113	0.01078	0.00552	0.00290	0.00153	0.18770	0.03079

5 Estimation of the Location Parameter of the Exponential Distribution with Known Coefficient of Variation by Record Values

A continuous random variable X is said to follow the exponential distribution with location parameter μ and scale parameter $a\mu$, if its pdf is given by,

$$f(x; \mu, a\mu) = \begin{cases} \frac{1}{a\mu} \exp\left\{-\frac{x-\mu}{a\mu}\right\}, & x \geq \mu, \mu > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

We will write $E(\mu, a\mu)$ to denote the exponential distribution defined in (18). In this case $E(X) = \mu + a\mu = (a + 1)\mu$, $Var(X) = a^2\mu^2$ and coefficient of variation $c = \frac{a}{a+1}$, where a is known positive constant. In this section we consider only upper record values and hence whenever we use the term record values in this section it means upper record values. The results based on record values arising from the two parameter exponential distribution $E(\mu, \sigma)$ are available in (see Arnold et al. [5], p.134). By using the results available in (see Arnold et al. [5], p.134), we have found out the following results,

$$\begin{aligned} \alpha' \mathbf{B}^{-1} \alpha &= n + 1, \quad \alpha' \mathbf{B}^{-1} \mathbf{1} = 1, \quad \mathbf{1}' \mathbf{B}^{-1} \mathbf{1} = 1, \\ \alpha' \mathbf{B}^{-1} &= (0, 0, \dots, 1) \quad \text{and} \quad \mathbf{1}' \mathbf{B}^{-1} = (1, 0, \dots, 0). \end{aligned}$$

Using the above results, the BLUE corresponding to (9) for the parameter μ involved in (18) reduces to,

$$\tilde{\mu}_1 = \frac{2 + n(n+1)a}{2[a^2(n+1) + 2a + 1]} R + \frac{a}{a^2(n+1) + 2a + 1} R_n,$$

that is,

$$\tilde{\mu}_1 = g_1 R + g_2 R_n, \quad (19)$$

where $g_1 = \frac{2+n(n+1)a}{2[a^2(n+1)+2a+1]}$, $g_2 = \frac{a}{a^2(n+1)+2a+1}$ and $R = R_0 + R_1 + \dots + R_n$.

And its variance reduces to,

$$Var(\tilde{\mu}_1) = \frac{a^2 \mu^2}{a^2(n+1) + 2a + 1}, \tag{20}$$

Table 3. Coefficients of R , coefficients of R_n in the BLUE $\tilde{\mu}_1$, $V_1 = \frac{Var(\tilde{\mu}_1)}{\mu^2}$, for different value for c

n	c	g_1	g_2	V_1
2	0.15	1.05742	0.12201	0.02153
3		1.39344	0.11944	0.02108
4		1.83257	0.11697	0.02064
5		2.36854	0.11461	0.02022
6		2.99559	0.11233	0.01982
7		3.70842	0.11015	0.01944
8		4.50212	0.10805	0.01907
9		5.37214	0.10603	0.01871
10		6.31429	0.10408	0.01873
2		0.2	1.03704	0.14815
3	1.42857		0.14286	0.03571
4	1.93103		0.13793	0.03448
5	2.53333		0.13333	0.03333
6	3.22581		0.12903	0.03226
7	4.00000		0.12500	0.03125
8	4.84848		0.12121	0.03030
9	5.76471		0.11765	0.02941
10	6.74286		0.11429	0.02857
2	0.25		1.00000	0.16667
3		1.42195	0.15789	0.05263
4		1.95000	0.15000	0.05000
5		2.57143	0.14286	0.04762
6		3.27273	0.13636	0.04545
7		4.04348	0.13043	0.04348
8		4.87500	0.12500	0.04167
9		5.76000	0.12000	0.04000
10		6.69231	0.11538	0.03846
2		0.3	0.94915	0.17797
3	1.37795		0.16535	0.07087
4	1.90441		0.15441	0.06618
5	2.51034		0.14483	0.06207
6	3.18182		0.13636	0.05844
7	3.90798		0.12883	0.05521
8	4.68023		0.12309	0.05233
9	5.49171		0.11602	0.04972
10	6.33684		0.11053	0.04737

where $a = \frac{c}{1-c}$, c is the known coefficient of variation. The main advantage of this results given in (19) and (20) is that, one can obtain the BLUE and its variance of the location parameter μ of the exponential distribution with known coefficient of variation by record values without knowing

the values of means, variances and covariances of the record values arising from the standard form of the distribution defined in (18). We have evaluated the coefficients of R , R_n in the BLUE $\tilde{\mu}_1$ defined in (19), $Var(\tilde{\mu}_1)$ defined in (20), for $n = 2(1)10$ and $c = 0.15(0.05)0.30$ and are given in Table 3.

6 Real Life Example

Roberts [32] has given the one-hour average concentration of sulphur dioxide (in pphm) from Long Beach, California, for the years 1956 to 1974. From this data, we observe the upper record values for the months July and August are respectively given by (14, 18, 37) and (21, 25, 26, 40, 55). Also the upper record values for the month of September to be (33, 38). For analysing the data of September based on upper record values, we have only two record values. Obviously we know that, for analysing a data based on only two values, it is not enough to get good result. This is a major drawback of record values, that is, statistical inference based on records arise due to the fact that the occurrences of record data are very rare in practical situations and moreover the expected waiting time is infinite for every record after the first. In this case, if any prior information can be incorporated, the result will be improved. The coefficient of variation of record values, for the months July and August are respectively given by 0.43 and 0.40 respectively. The coefficient of variation is a stable measure of dispersion and thus does not change quite rapidly over the years. Here also we can observe that the coefficient of variation is constant and is approximately equal to 0.40. This is a situation where one can make use of the past data to have a knowledge regarding coefficient of variation.

A simple plot of the upper record values of each month against the expected values arising from usual standard normal distribution given in Table 1 of Balakrishnan and Chan [30] indicate a very strong correlation (that is, the correlation coefficient between expected values and record values of each months as high as 0.90). Hence the assumption that the record values have come from a normal distribution $N(\mu, c\mu)$ is quite reasonable. From Table 1, we then determine the best linear unbiased estimator (BLUE) of μ to be,

$$\begin{aligned}\tilde{\mu} &= -0.03 \times 33 + 0.76 \times 38 \\ &= 27.89\end{aligned}$$

and

$$Var(\tilde{\mu}) = 0.06726.$$

Once again without loss of generality assume that the record values for the month of September arising from a logistic distribution $LD(\mu, c\mu)$. From Table 3, we then determine the best linear unbiased estimator (BLUE) of μ to be,

$$\begin{aligned}\tilde{\mu} &= -0.14 \times 33 + 0.84 \times 38 \\ &= 27.30\end{aligned}$$

and

$$Var(\tilde{\mu}) = 0.07701.$$

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Competing Interests

Author has declared that no competing interests exist.

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The Differential Transform Method (DTM): Solution of Differential Equations

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Abstract

In this chapter, linear and nonlinear differential equations are solved. The calculations are carried out by using differential transformation method (DTM) which is a semi-numerical–analytical solution technique. By using DTM, the nonlinear constrained governing equations are reduced to recurrence relations and related initial conditions are transformed into a set of algebraic equations. The properties of differential transformation is briefly introduced, and then applied for the number of problems. The current results are then compared with those derived from the classical Runge-Kutta method (RK4) order to verify the accuracy of the proposed method. The findings disclose that the DTM can achieve more suitable results in predicting the solution of such problems.

Keywords: Riccati equation; duffing equation; integro-differential equations; chaotic solution; differential transforms method; Runge-Kutta 4 (RK4) method.

1 Introduction

Differential equations have been solved by a variety of exact, approximate and purely numerical methods over these decades which need intensive computations. Wazwaz [1,2] used the Adomian Decomposition method to handle some system of partial differential equations and the reaction diffusion Brusselator model. On the other hand Batiha [3] improved the results obtained by Wazwaz by the application of Variational Iteration method to solve some linear and nonlinear systems of partial differential equations. Another method which is equally effective in solving various linear and nonlinear ordinary differential equations and partial differential equations is the Differential Transform method (DTM). This method (DTM), which is based on Taylor series expansion, was introduced by Zhou [4] in a study of electrical circuits. Later Chiou [5] applied this method to solve nonlinear vibration problems and Jang [6] applied the two-dimensional Differential Transform method to find the solution of various partial differential equations. Hassan [7] adopted the Differential Transformation method to solve some eigenvalue problems also. Not only differential equations but some difference equations, differential-difference equations and some boundary value problems for integro-differential equations have also been solved using differential transform method by Arikoglu and Ozkol [8,9,10]. Other systems like KdV equations, mKdV equations [11], Sine Gordon equations [12], Burgers equation [13], Emden type equations [14], nonlinear Klein Gordon equation [15], nonlinear Gas Dynamic equation, some wave and heat conduction equations etc. have also been solved by this method and it has been proved that the results thus obtained are in good agreement with the exact solutions and also with the solutions obtained by some numerical methods like RungeKutta fourth-order method. Some chaotic systems have been solved by this method like LotkaVolterra, Chen and Lorenz systems [16].

This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor-series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor-series method is computationally expensive for large orders, whereas the DTM is an iterative procedure for obtaining an analytical Taylor-series solution of a differential equation. The differential transform has the inherent ability to deal with nonlinear problems also. An implication of the DTM is that only a limited number of terms are required to be summed and therefore the solution can be computed rapidly. The

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method gives more satisfactory results for small times. However we may obtain better results by adding more terms to the right-hand side of the DTM solution for longer time intervals.

2 One Dimensional Differential Transform

Let $f(x)$ be an analytic function in a domain D . The differential transform (DT) of $f(x)$ is given by

$$F(k) = \left. \frac{1}{k!} \frac{d^k}{dx^k} [f(x)] \right|_{at\ x=x_0} \quad \text{at } x_0 \in D, \quad k = 0, 1, 2, 3, \dots \text{ being the transform parameter.} \quad (2.1)$$

where $F(k)$ is the transformed function of $f(x)$.

The function $f(x)$ is obtained from the inverse differential transform defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k.$$

From equation (2.1) we may write $f(x) = \sum_{k=0}^{\infty} (x-x_0)^k \left. \frac{1}{k!} \frac{d^k}{dx^k} [f(x)] \right|_{at\ x=x_0}$.

The particular case when $x_0 = 0$ is referred to as the Maclaurin series and is expressed as

$$f(x) = \sum_{k=0}^{\infty} x^k \left. \frac{1}{k!} \frac{d^k}{dx^k} [f(x)] \right|_{at\ x=0}$$

This clearly shows that the concept of differential transform is derived from Taylor's series expansion. However the method does not require to evaluate the derivatives rigorously. This method is a semi analytic numerical method where the function is approximated by a series, the approximation being implemented as

$$f(x) \approx \sum_{k=0}^N F(k)(x-x_0)^k \quad (2.2)$$

Here N is decided by the convergence of natural frequency.

The differential transform of some fundamental function at $x = 0$ are given below:

1. When $f(x) = x$ then

$$DT[f(x)] = \left. \frac{1}{k!} \frac{d^k}{dx^k} [x] \right|_{at\ x=0} = \delta(k-1) \quad \text{where } \delta(k-1) = \begin{cases} 1 & \text{when } k=1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{When } f(x) = x^m \text{ then } DT[f(x)] = \delta(k-m) \text{ where } \delta(k-m) = \begin{cases} 1 & \text{when } k=m \\ 0 & \text{elsewhere} \end{cases}$$

2. When $f(x) = u(x) \pm v(x)$ then

$$DT[f(x)] = \left. \frac{1}{k!} \frac{d^k}{dx^k} [u(x) \pm v(x)] \right|_{at\ x=0} = U(k) \pm V(k) \quad \text{where } U(k) \text{ and } V(k) \text{ are the differential}$$

transform of $u(x)$ and $v(x)$ respectively.

3. When $f(x) = cu(x)$ then

$$DT[f(x)] = \frac{1}{k!} \frac{d^k}{dx^k} [cu(x)] \Big|_{at\ x=0} = c \frac{1}{k!} \frac{d^k}{dx^k} [u(x)] \Big|_{at\ x=0} = cU(k)$$

where $U(k)$ is the differential transform of $u(x)$.

4. When $f(x) = u(x)v(x)$ then $DT[f(x)] = \frac{1}{k!} \frac{d^k}{dx^k} [u(x)v(x)] \Big|_{at\ x=0}$

Applying Leibnitz Rule

$$\begin{aligned} & \frac{1}{k!} \frac{d^k}{dx^k} [u(x)v(x)] \Big|_{at\ x=0} \\ &= \frac{1}{k!} \left[v(x) \frac{d^k u(x)}{dx^k} + k \frac{dv(x)}{dx} \frac{d^{k-1} u(x)}{dx^{k-1}} + \frac{k(k-1)}{2!} \frac{d^2 v(x)}{dx^2} \frac{d^{k-2} u(x)}{dx^{k-2}} + \dots \right. \\ & \quad \left. + \dots + \frac{1}{(k-1)!} \frac{d^{k-1} v(x)}{dx^{k-1}} \frac{du(x)}{dx} + \frac{1}{(k)!} \frac{d^k v(x)}{dx^k} u(x) \right] \Big|_{at\ x=0} \end{aligned}$$

$$DT[f(x)] = V(0)U(k) + V(1)U(k-1) + \dots + V(k-1)U(1) + V(k)U(0)$$

$$= \sum_{l=0}^k V(l)U(k-l)$$

where $U(k)$ and $V(k)$ are the differential transform of $u(x)$ and $v(x)$ respectively.

If $f(x) = u(x)v(x)w(x)$, then $F(k) = \sum_{s=0}^k \sum_{m=0}^{k-s} U(s)V(m)W(k-s-m)$.

Where $F(k), G(k), H(k), U(k), V(k), W(k)$ are the differential transform of the functions $f(x), g(x), h(x), u(x), v(x), w(x)$ respectively.

5. When $f(x) = \frac{du(x)}{dx}$ then

$$\begin{aligned} DT[f(x)] &= \frac{1}{k!} \frac{d^k}{dx^k} \left[\frac{du(x)}{dx} \right] \Big|_{at\ x=0} = \frac{1}{k!} \frac{d^{k+1} u(x)}{dx^{k+1}} \Big|_{at\ x=0} \\ &= \frac{(k+1)!}{k!} \frac{1}{(k+1)!} \frac{d^{k+1} u(x)}{dx^{k+1}} \Big|_{at\ x=0} = (k+1)U(k+1) \end{aligned}$$

where $U(k)$ is the differential transform of $u(x)$.

6. When $f(x) = \int_{x_0}^x u(t)dt$ then

$$DT[f(x)] = \frac{1}{k!} \frac{d^k}{dx^k} \left[\int_{x_0}^x u(t)dt \right] \Big|_{at\ x=0} = \frac{1}{k!} \frac{d^{k-1}}{dx^{k-1}} \left[\frac{d}{dx} \left(\int_{x_0}^x u(t)dt \right) \right] \Big|_{at\ x=0}$$

Applying Leibnitz Rule of Differentiation under the sign of Integration

$$= \frac{1}{k!} \frac{d^{k-1}u(x)}{dx^{k-1}} \Big|_{at\ x=0} = \frac{1}{k} U(k-1), \quad k \geq 1.$$

where $U(k)$ is the differential transform of $u(x)$.

7. When $f(x) = \exp(\lambda x)$ then $DT[f(x)] = \frac{1}{k!} \frac{d^k}{dx^k} [\exp(\lambda x)] \Big|_{at\ x=0} = \frac{\lambda^k}{k!}$

Applying differential transform to a differential equation in $f(x)$ (say) we obtain a difference equation which is used to obtain the transform function for different values of the transform parameter. These are then used in (2.2) to obtain $f(x)$. The following examples help us to understand the method.

3 Examples

3.1 Solution of the Riccati equation with variable co-efficients

The Riccati equation with variable co-efficients is given as

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2 \tag{3.1.1a}$$

$$y(0) = g(x) \tag{3.1.1b}$$

where $p(x)$, $q(x)$, $r(x)$ and $g(x)$ are functions of x . Applying Differential Transform (DT) to the above equation we have

$$DT \left[\frac{dy}{dx} = p(x) + q(x)y(x) + r(x)y^2(x) \right] \\ \Rightarrow (k+1)T(k+1) = P(k) + \sum_{l=0}^k Q(l)T(k-l) + \sum_{s=0}^k \sum_{m=0}^{k-s} R(s)T(m)T(k-s-m) \tag{3.1.2}$$

where $P(k)$, $Q(k)$, $R(k)$ and $T(k)$ are the differential transforms of $p(x)$, $q(x)$, $r(x)$ and $y(x)$. The above general equation (3.1.1a,b) when $p(x) = x^5 + 1$, $q(x) = -2x^4$, $r(x) = x^3$ and $g(x) = 0$ takes the form

$$\frac{dy}{dx} = 1 + x^5 - 2x^4y(x) + x^3y^2(x), \tag{3.1.3a}$$

$$\text{with initial condition as } y(0) = 0 \tag{3.1.3b}$$

Applying Differential Transform (DT) to (3.1.3a) we have

$$\begin{aligned} \text{DT} \left[\frac{dy}{dx} = 1 + x^5 - 2x^4 y(x) + x^3 y^2(x) \right], \\ \Rightarrow \text{DT} \left[\frac{dy}{dx} \right] = \text{DT}[1] + \text{DT}[x^5] - 2\text{DT}[x^4 y] + \text{DT}[x^3 y^2(x)] \end{aligned} \quad (3.1.4)$$

$$\Rightarrow (k+1)T(k+1) = \delta(k) + \delta(k-5) - 2 \sum_{l=0}^k \delta(l-4)T(k-l) - \sum_{s=0}^k \sum_{m=0}^{k-s} \delta(s-3)T(m)T(k-s-m) \quad (3.1.5)$$

The inverse differential transform of $y(x)$ is $y(x) = \sum_{k=0}^{\infty} T(k)x^k$ (3.1.6)

Using the initial condition $y(0) = 0$ we have, $T(0) = 0$ (3.1.7)

For $k=0$ in the above equation and using equation (3.1.5), we have

$$T(1) = \delta(0) = 1. \quad (3.1.8)$$

$$\text{For } k=1, \text{ we have } 2T(2) = 0 \Rightarrow T(2) = 0. \quad (3.1.9)$$

For $k=2$, we have $3T(3) = \sum_{s=0}^2 \sum_{m=0}^{2-s} \delta(s-3)T(m)T(2-s-m) - 2 \sum_{l=0}^2 \delta(l-4)T(2-l)$,
 $\Rightarrow T(3) = 0.$ (3.1.10)

Similarly for $k=3, 4, 5, 6, 7, 8$ and 9 we have

$$T(4) = T(5) = T(6) = \dots = 0, \quad (3.1.11)$$

From equation (3.1.4) we have the exact solution of the equation (3.1.3a) given as

$$y(x) = x \quad (3.1.12)$$

3.2 Solution of unforced damped duffing equation

$$\begin{aligned} \frac{d^2 x}{dt^2} + \delta \frac{dx}{dt} + \beta x + \varepsilon x^3 = 0, \\ x(0) = a, x'(0) = 0 \end{aligned}$$

Using DTM

$$X(k+2) = \frac{1}{(k+1)(k+2)} \left[-\delta(k+1)X(k+1) - \beta X(k) - \varepsilon \sum_{m=0}^k \sum_{n=0}^{k-m} X(m)X(n)X(k-m-n) \right] = 0$$

and $X(0) = a$ and $X(1) = 0$, $X(k)$ being the transform function of $x(t)$.

From the inverse differential transform we have,

$$x(t) = a - \frac{a}{2}(\varepsilon a^2 + \beta)t^2 + \frac{a\delta}{6}(\varepsilon a^2 + \beta)t^3 - \frac{a}{24}(\varepsilon a^2 + \beta)(3\varepsilon a^2 + \beta - \delta^2)t^4 + \dots$$

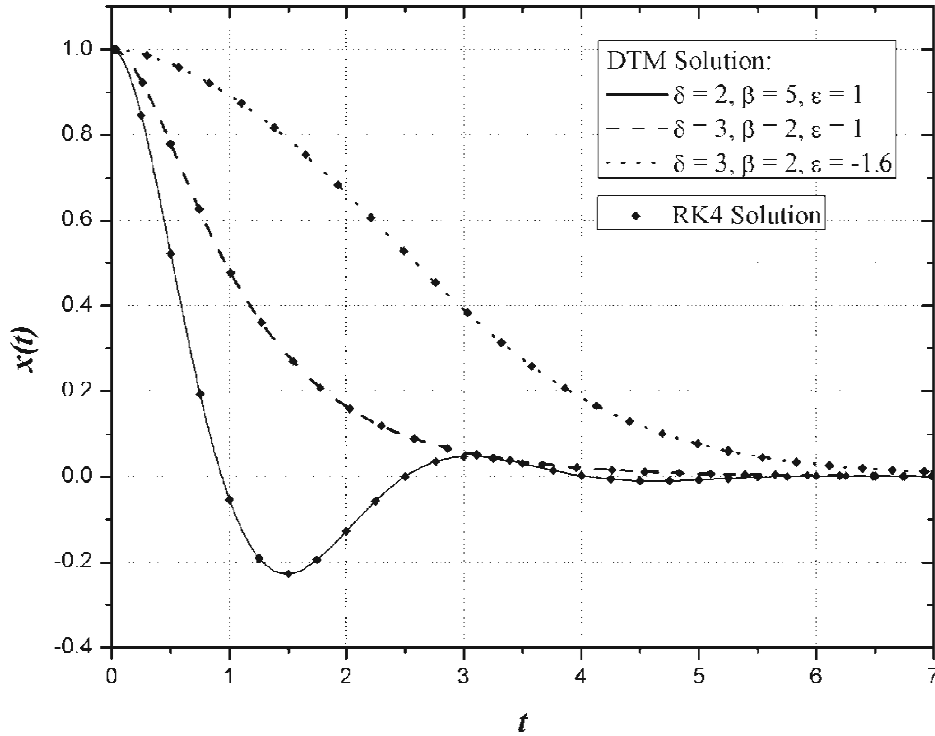


Fig. 1. Plot of the comparison between solutions obtained by DTM and RK4 of unforced damped duffing equation for different values of parameters with initial condition $x(0) = 1, x'(0) = 0$

3.3 Solution of forced damped duffing equation

$$\frac{d^2x}{dt^2} + \delta \frac{dx}{dt} + \beta x + \epsilon x^3 = F \sin \omega t$$

subject to the initial conditions:

$$x(0) = a, \quad x'(0) = 0$$

Using DTM,

$$X(k+2) = \frac{1}{(k+1)(k+2)} \left[F \frac{\omega^k}{k!} \sin\left(\frac{k\pi}{2}\right) - \delta(k+1)X(k+1) - \beta X(k) - \epsilon \sum_{m=0}^k \sum_{n=0}^{k-m} X(m)X(n)X(k-m-n) \right] = 0$$

and the initial conditions are given by $X(0) = a$ and $X(1) = 0$. It should be noted that if $F = 0$ i.e. for the unforced case, $X(k+2)$ coincides. Moreover, it is observed that for the even values of the transform variable k , two solutions are same.

From the inverse differential transform,

$$x(t) = a - \frac{a}{2}(\epsilon a^2 + \beta)t^2 + \left[\frac{a\delta}{6}(\epsilon a^2 + \beta) + \frac{F\omega}{6} \right] t^3 - \left[\frac{a}{24}(\epsilon a^2 + \beta)(3\epsilon a^2 + \beta - \delta^2) - \frac{F\omega\delta}{24} \right] t^4 + \dots$$

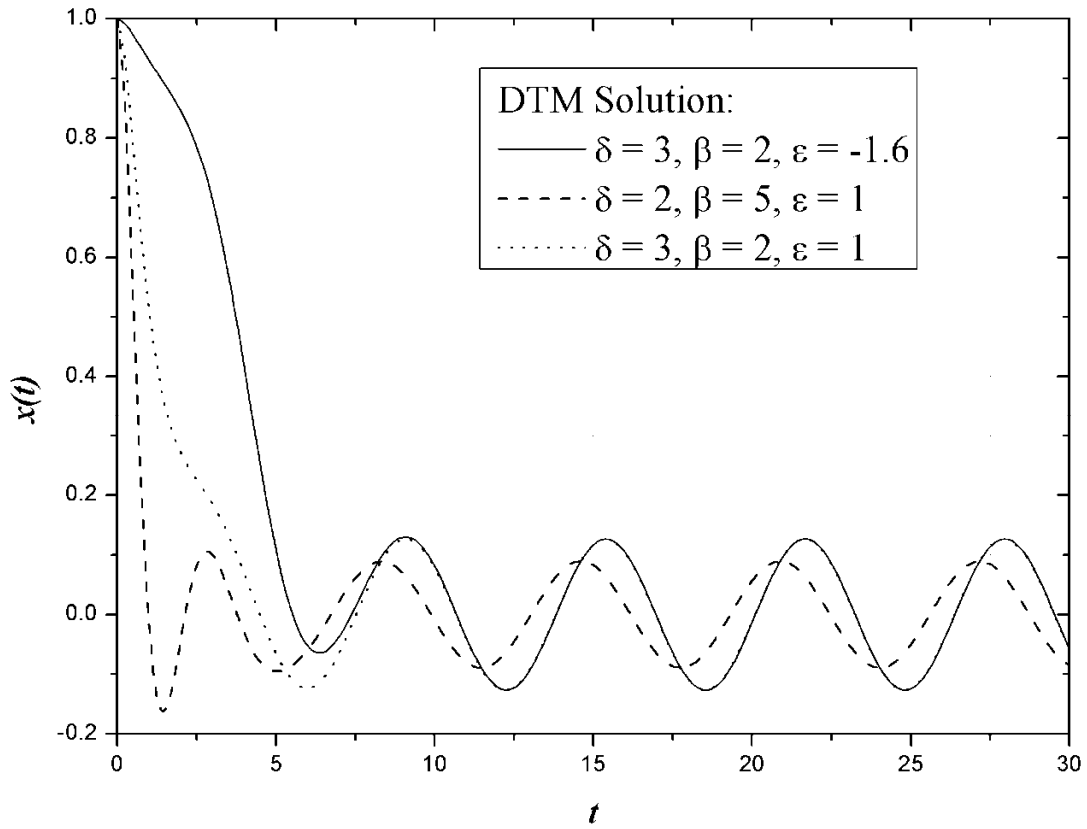


Fig. 2. Plot of the solution of forced damped duffing equation obtained by DTM for different values of parameters with initial condition $x(0) = 1, x'(0) = 0$

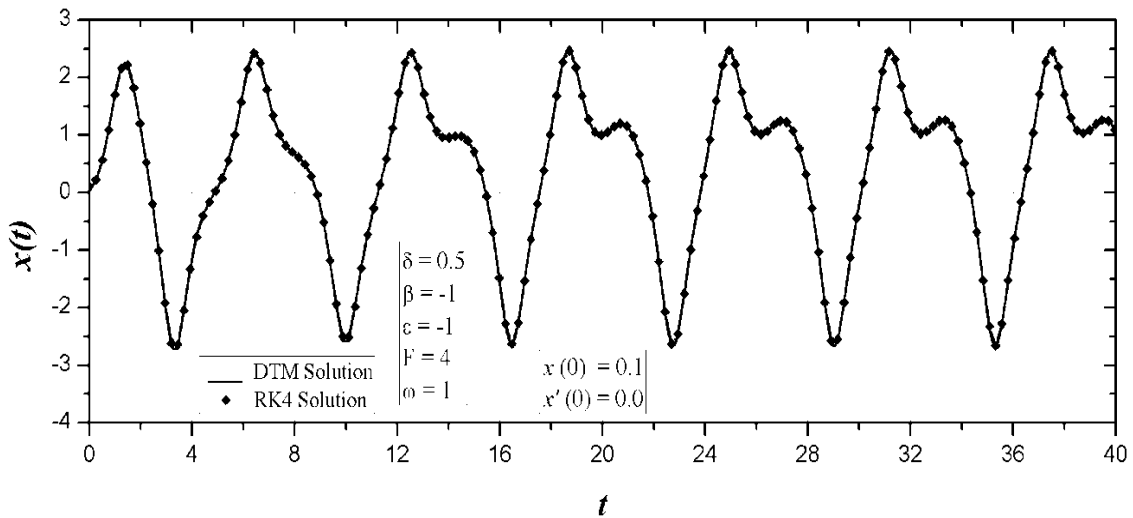


Fig. 3. Plot of the comparison between solutions obtained by DTM and RK4 of forced damped duffing equation for different values of parameters with initial condition $x(0) = 1, x'(0) = 0$

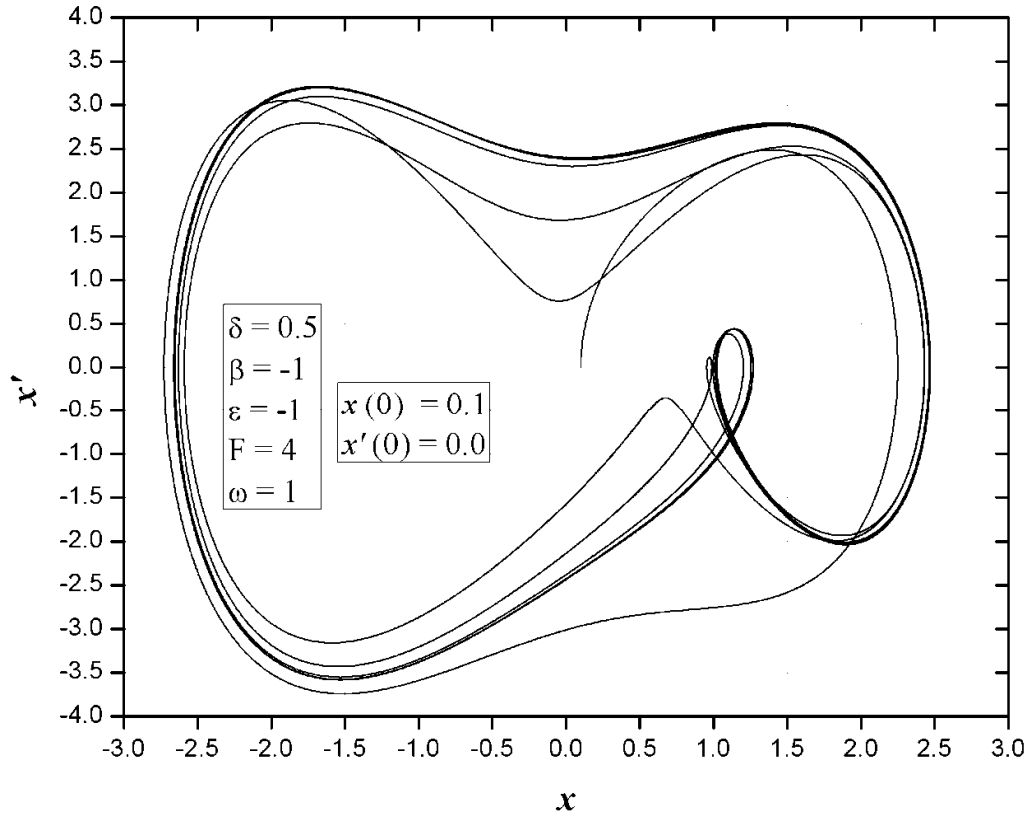


Fig. 4. Phase plot of forced damped duffing equation

However if we consider the time interval to be $(0, H)$ then the differential transform is redefined as

$$F(k) = \frac{H}{k!} \left. \frac{d^k}{dx^k} [f(x)] \right|_{at \ x=0} \quad k = 0, 1, 2, \dots \text{ being the transform parameter.}$$

where $F(k)$ is the transformed function of $f(x)$.

The function $f(x)$ is obtained from the inverse differential transform defined as

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{H} \right)^k F(k).$$

3.4 Solution of three-dimensional quadratic autonomous system

The three-dimensional quadratic autonomous system is given by:

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{ab}{a+b}x(t) - y(t)z(t) + c \\ \frac{dy(t)}{dt} = ay(t) + x(t)z(t) \\ \frac{dz(t)}{dt} = bz(t) + x(t)y(t) \end{cases} \quad (3.4.1)$$

subject to the initial conditions:

$$x(0) = 1, y(0) = 1, z(0) = 1, \tag{3.4.2}$$

where a, b, c are real constants.

Applying DTM to equation (3.4.1), the series solution for the system is given by,

$$X(k+1) = \frac{H}{k+1} \left[-\frac{ab}{a+b} X(k) - \sum_{l=0}^k Y(l)Z(k-l) + c \right], \tag{3.4.3}$$

$$Y(k+1) = \frac{H}{k+1} \left[aY(k) + \sum_{l=0}^k X(l)Z(k-l) \right], \tag{3.4.4}$$

$$Z(k+1) = \frac{H}{k+1} \left[bZ(k) + \sum_{l=0}^k X(l)Y(k-l) \right], \tag{3.4.5}$$

where $X(k)$, $Y(k)$ and $Z(k)$ are the differential transforms of the functions $x(t)$, $y(t)$ and $z(t)$ respectively and $0 \leq t \leq H$, H being a constant. The equations (3.4.3) – (3.4.5) represent the difference equations for the system under consideration. From the inverse differential transform, we obtain the solutions in the time domain which are given by:

$$x_p(t) = \sum_{k=0}^n \left(\frac{t}{H_p} \right)^k X_p(k), \quad t \in [0, H_p], \tag{3.4.6}$$

$$y_p(t) = \sum_{k=0}^n \left(\frac{t}{H_p} \right)^k Y_p(k), \quad t \in [0, H_p], \tag{3.4.7}$$

$$z_p(t) = \sum_{k=0}^n \left(\frac{t}{H_p} \right)^k Z_p(k), \quad t \in [0, H_p]. \tag{3.4.8}$$

Here $k = 0, 1, 2, \dots, n$ represents the number of terms of the power series and H_p denotes the length of p^{th} sub-domain interval with $p = 0, 1, 2, \dots$

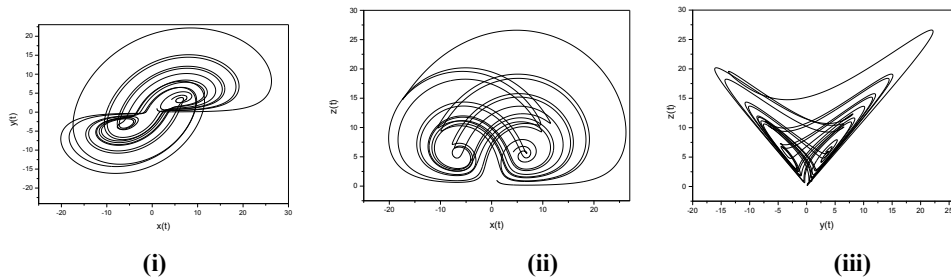


Fig. 5. Chaotic attractors for the three-dimensional quadratic autonomous system (1) when $(a, b, c) = (-10, -4, 0)$ obtained by DTM, at step size $h = 0.001$ and $H = 25$

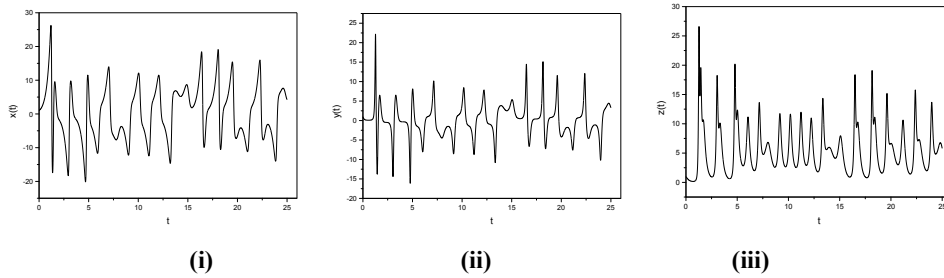


Fig. 6. Time series analysis for the three-dimensional quadratic autonomous system (1) when $(a, b, c) = (-10, -4, 0)$ obtained by DTM, at step size $h = 0.001$ and $H = 25$

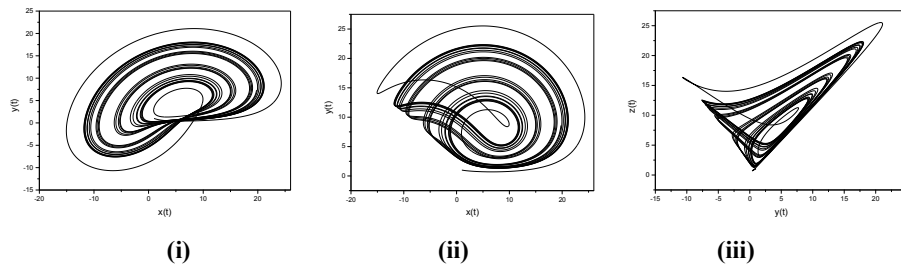


Fig. 7. Chaotic attractors for the three-dimensional quadratic autonomous system (1) when $(a, b, c) = (-10, -4, 18.1)$ obtained by DTM, at step size $h = 0.001$ and $H = 25$

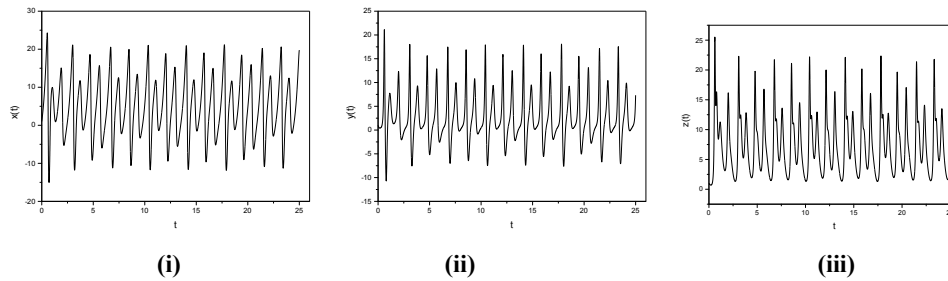


Fig. 8. Time series analysis for the three-dimensional quadratic autonomous system (3.4.1) when $(a, b, c) = (-10, -4, 18.1)$ obtained by DTM, at step size $h = 0.001$ and $H = 25$

3.5 Solution of integro-differential equation

The method of differential transform may be used to solve integro-differential equations.

Let us consider the integro-differential equation

$$y'(x) = 1 - \int_0^x y(t) dt ; \quad y(0) = 0 \tag{3.5.1}$$

Applying DTM to the equation (3.5.1) we have the recurrence relation as

$$(k+1)Y(k+1) = \delta(k) - \frac{Y(k-1)}{k} \Rightarrow k(k+1)Y(k+1) = k\delta(k) - Y(k-1) \tag{3.5.2}$$

where $Y(k)$ is the differential transform of the function $y(x)$,

$$\text{For } k = 1, \quad 2Y(2) = \delta(1) - Y(0) \Rightarrow Y(2) = 0$$

$$\text{For } k = 2, \quad 3!Y(3) = 2\delta(2) - Y(1) \Rightarrow Y(3) = -\frac{Y(1)}{3!}$$

$$\text{For } k = 3, \quad 3 \times 4Y(4) = 3\delta(3) - Y(2) \Rightarrow Y(4) = 0$$

$$\text{For } k = 4, \quad 4 \times 5Y(5) = 4\delta(4) - Y(3) \Rightarrow Y(5) = -Y(3) \Rightarrow Y(5) = \frac{Y(1)}{5!} \text{ and so on.}$$

$$\text{From the inverse differential transform } y(x) = \sum_{k=0}^{\infty} Y(k)x^k \tag{3.5.3}$$

and initial condition $y(0) = 0$ we have $Y(0) = 0$. From equation (3.5.1) we have $y'(0) = 1 \Rightarrow Y(1) = 1$.

Thus from (3.5.3) $y(x) = Y(0) + xY(1) + x^2Y(2) + x^3Y(3) + \dots$

$$\Rightarrow y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \Rightarrow y(x) = \sin x.$$

4 Conclusion

The proposed method may be considered to be an efficient and accurate method that can be used to provide analytical solutions for various types of differential, integral and integro differential equations which are linear and nonlinear, homogeneous and non-homogeneous, with constant and with variable coefficients. DTM proves to be an important tool in handling highly nonlinear differential equations. It requires a minimum size of computations and has a wide interval of convergence. The straightforward applicability, computational effectiveness and the accuracy of the results obtained by DTM is evident from the examples stated above, where exact analytical solutions may also be obtained by this method. This method provides the solution in a rapidly convergent series with components that are computed both elegantly and accurately. The main advantage of the method is that it reduces the size of computation work and at the same time maintains a high level of accuracy. This method may also be applied in solving two dimensional differential equations [15].

Competing Interests

Authors have declared that no competing interests exist.

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Solution of Modified Equations of Emden-type by Differential Transform Method: New Perspectives

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Abstract

In this paper the Modified Equations of Emden type (MEE), $\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0$ is solved numerically by the differential transform method. This technique doesn't require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computation. The current results of this paper are in excellent agreement with those provided by Chandrasekar et al. [1] and thereby illustrate the reliability and the performance of the differential transform method. We have also compared the results with the classical Runge-Kutta 4 (RK4) Method.

Keywords: Modified equations of Emden type; differential transforms method; Runge-Kutta 4 (RK4) method.

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1 Introduction

The modified equation of Emden type (MEE), also called the modified Painleve-Ince equation,

$$\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0 \quad (1)$$

where over dot denotes differentiation with respect to time and α and β are arbitrary parameters, have received attention from both mathematicians and physicists for more than a century [2-6]. The above differential equation appears in a number of mathematical problems such as univalued functions defined by second order differential equations [7] and the Riccati equation [8]. Physicists have found this equation in the study of equilibrium configurations of a spherical gas cloud acting under the mutual attraction of its molecules and subject to the laws of thermodynamics [9,10], in spherically symmetric expansion or collapse of a relativistic gravitating mass [11] and in the modeling of the fusion of pellets [12]. The invariance and the integrability of this equation have been a subject of study for the past two decades by a number of authors [13-23]. This equation have been found to possess an explicit general solution for the following parametric choices,

$$(i) \alpha = 0, \quad (2a)$$

$$(ii) \beta = 0, \quad (2b)$$

$$(iii) \beta = \frac{\alpha^2}{9} \quad (2c)$$

$$(iv) \beta = -\alpha^2 \quad (2d)$$

However, the general solution of equation (1) for arbitrary values of α and β was explored for the first time by Chandrasekhar et al. [1]. They have constructed the time-independent Hamiltonians from the time-independent integrals of equation (1) and by the suitable use of canonical transformations, have converted these

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Hamiltonians to their standard forms. The general solutions are then obtained by integrating these new Hamiltonians. We present here a humble effort to arrive at the same by the Differential Transform Method [DTM].

The concept of differential transform was first introduced by Zhou [24] in solving linear and nonlinear initial value problems in electrical circuit analysis. The traditional Taylor series method takes a long time for computation of higher order derivatives. Instead, DTM is an iterative procedure for obtaining analytic Taylor series solution of differential equations and is much easier. In our previous work we have seen that the DTM provides the solution of the Duffing-Van der Pol oscillator equation in a rapidly convergent series [25] and that, it is in good agreement with the solution obtained by Chandrasekar et al. [26].

2 The Modified Emden-Type Equations

As already mentioned, the modified equation of Emden type cannot be integrated straightforwardly for arbitrary values of α and β . The solution of MEE for the particular choice of parameters given by (2a) and (2b) can be obtained by simple integration and for the choice (2c), the equation is linearizable to a free particle equation. In the fourth case the general solution can be expressed in terms of the Weierstrass elliptic function [2-6,13-23,27]. It has also been noted that the MEE possess the Painleve` property for certain values of $r = \left(\frac{\alpha}{4\beta}\right) (\alpha \pm \sqrt{\alpha^2 - 8\beta})$ [17,19].

In [1] the authors have identified the first integrals of equation (1) separately for each of the three ranges (i) $\alpha^2 = 8\beta$, (ii) $\alpha^2 > 8\beta$, and (iii) $\alpha^2 < 8\beta$. The Hamiltonians are obtained from these integrals and are given by

$$H = \begin{cases} \log\left(\frac{-4\alpha}{p}\right) - \frac{\alpha}{4}px^2 \dots \dots \dots \alpha^2 = 8\beta \\ \frac{r-1}{r-2}p^{\frac{r-2}{r-1}} - \frac{r-1}{2r}\alpha x^2 p \dots \dots \dots \alpha^2 > 8\beta \\ \frac{1}{2}\log\left[x^4 \sec^2\left(\frac{\omega}{4}x^2 p\right)\right] - \frac{\alpha}{4}x^2 p \dots \dots \alpha^2 < 8\beta. \end{cases} \quad (3)$$

For the case $\alpha^2 = 8\beta$ the Hamiltonian $H = \log\left(\frac{-4\alpha}{p}\right) - \frac{\alpha}{4}px^2$ reduces to the standard form

$$H = \frac{1}{2}p^2 + \log\left(\frac{32}{U^2}\right), \quad (4)$$

under the canonical transformation

$$x = \frac{4P}{\alpha U}, p = -\frac{\alpha U^2}{8} \quad (5)$$

The general solution thus obtained by integrating the new Hamiltonian (4) and by using the canonical equations $\dot{U} = P$ and $\dot{P} = \frac{2}{U}$ is given by,

$$x(t) = \frac{8}{i\alpha} \operatorname{erf}^{-1}(z) \exp\left[\frac{1}{2}(E + 2[\operatorname{erf}^{-1}(z)]^2)\right] \quad (6)$$

where, $z = \frac{\{2(t_0 + it) \exp[E/2]\}}{\sqrt{\pi}}$, $E = \frac{1}{2}\dot{U}^2 - 2 \log U$, t_0 is an arbitrary constant of integration and erf is the error function [28].

In our present work we have solved the modified equation of Emden type by the Differential transform method and we have compared the results with equation (6) [1]. We have also compared the results with those obtained by Runge-Kutta 4 Method.

3 The Differential Transform Method

Differential transform of a function $f(x)$ is defined as follows

$$F(k) = \frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=0}. \quad (7)$$

In (7), $f(x)$ is the original function and $F(k)$ is the transformed function. The Taylor series expansion of the function $f(x)$ about a point $x = 0$ is given as

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=0}.$$

Replacing $\frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=0}$ by $F(k)$, we have

$$f(x) = \sum_{k=0}^{\infty} x^k F(k) \tag{8}$$

which may be defined as the inverse differential transform.

From (7) and (8) it is easy to obtain the following mathematical operations:

- i. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$
- ii. If $f(x) = cg(x)$, then $F(k) = cG(k)$, where c is a constant.
- iii. If then $f(x) = \frac{d^n g(x)}{dx^n}$, then $F(k) = \frac{(k+n)!}{k!} G(k+n)$.
- iv. If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^k G(l)H(k-l)$.
- v. If $f(x) = x^n$, then $F(k) = \delta(k-n)$, where δ is Kronecker delta.
- vi. If $f(x) = \int_0^x g(t)dt$, then $F(k) = \frac{G(k-1)}{k}$, where $k \geq 1$.
- vii. If $f(x) = u(x)v(x)w(x)$, then $F(k) = \sum_{s=0}^k \sum_{m=0}^{k-s} U(s)V(m)W(k-s-m)$.

Where $F(k), G(k), H(k), U(k), V(k), W(k)$ are the differential transform of the functions $f(x), g(x), h(x), u(x), v(x), w(x)$ respectively.

4 Solution of the Modified Emden-Type Equations Using Differential Transform Method

The equation of the modified Emden type is given as

$$\frac{d^2x}{dt^2} + \alpha x \frac{dx}{dt} + \beta x^3 = 0 \tag{9}$$

The initial conditions are $x(0) = 0$ and $x'(0) = 1$ (where prime denotes differentiation with respect to time).

Applying Differential Transform (DT) to (9), we have

$$DT \left[\frac{d^2x}{dt^2} + \alpha x \frac{dx}{dt} + \beta x^3 = 0 \right]$$

or

$$DT \left[\frac{d^2x}{dt^2} \right] + \alpha DT \left[x \frac{dx}{dt} \right] + \beta DT[x^3] = 0,$$

i.e.

$$(k+2)(k+1)T(k+2) + \alpha \sum_{l=0}^k T(l)(k+1-l)T(k+1-l) + \beta \sum_{s=0}^k \sum_{m=0}^{k-s} T(s)T(m)T(k-s-m) = 0. \tag{10}$$

The inverse differential transform of $T(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} T(k)t^k. \tag{11}$$

Using the initial conditions $x(0) = 0$; $\frac{dx(0)}{dt} = 1$ we have, $T(0) = 0$ and $T(1) = 1$.

For k=0 in the above equation, we have

$$2T(2) + \alpha T(0)T(1) + \beta T(0)T(0)T(0) = 0 \Rightarrow T(2) = 0. \quad (12)$$

For k=1, we have

$$6T(3) + \alpha[2T(0)T(2) + T(1)T(1)] + \beta[\sum_{m=0}^1 T(0)T(m)T(1-m) + T(1)T(0)T(0)] = 0, \\ \Rightarrow T(3) = \frac{-\alpha}{6}. \quad (13)$$

For k=2, we have

$$12T(4) + \alpha \sum_{l=0}^3 (3-l)T(l)T(3-l) + \beta \sum_{s=0}^2 \sum_{m=0}^{2-s} T(s)T(m)T(2-s-m) = 0, \\ \Rightarrow T(4) = 0. \quad (14)$$

For k=3, 4, 5, 6, 7, we have

$$T(5) = \frac{\alpha^2}{30} - \frac{\beta}{20}, \quad (15)$$

$$T(6) = 0, \quad (16)$$

$$T(7) = \frac{-17\alpha^3}{2520} + \frac{2\alpha\beta}{105}, \quad (17)$$

$$T(8) = 0, \quad (18)$$

$$T(9) = \frac{1395\alpha^4}{1020600} - \frac{379\alpha^2\beta}{30240} + \frac{3\beta^2}{1440}, \quad (19)$$

$$\text{and } T(10) = 0 \quad (20)$$

For the case $\alpha^2 = 8\beta$, from equations (11 to 19) we have,

$$T(2) = 0, \quad (21)$$

$$T(3) = -\frac{\alpha}{6}, \quad (22)$$

$$T(4) = 0, \quad (23)$$

$$T(5) = \frac{\alpha^2}{30} - \frac{\beta}{20} = \frac{13\alpha^2}{60}, \quad (24)$$

$$T(6) = 0, \quad (25)$$

$$T(7) = \frac{-17\alpha^3}{2520} + \frac{2\alpha\beta}{105} = \frac{-231\alpha^3}{52920}, \quad (26)$$

$$T(8) = 0, \quad (27)$$

$$T(9) = \left(\frac{1395}{1020600} - \frac{6025}{9072000} + \frac{3}{92160} \right) \alpha^4 = \frac{735}{1000000} \alpha^4, \quad (28)$$

$$T(10) = 0. \quad (29)$$

From equation (11) we have $x(t) = \sum_{l=0}^{\infty} t^l T(l)$

$$\Rightarrow x(t) = t - t^3 \frac{\alpha}{6} + t^5 \frac{13\alpha^2}{60} - t^7 \frac{231\alpha^3}{52920} + t^9 \frac{735\alpha^4}{1000000} - \dots \infty \quad (30)$$

5 Comparison of Results

The solution plot of the Modified Equations of Emden type using DTM is given in Fig. 1 for the parametric choice $\alpha^2 = 8\beta$ for different values of α . The graphical representation of the solution (30) obtained by the DTM in this paper is in good agreement with those obtained by Chandrasekar et al. [1] and thereby illustrate the reliability and the performance of the differential transform method. Fig. 2 gives us a comparison of the solution for MEE obtained by DTM with the solution obtained by classical Runge-Kutta 4 Method. Table 1 gives the estimate of absolute error between the DTM- solutions with RK4 solutions. It is clear from Fig. 1, Fig. 2 and Table 1, that the solution obtained by DTM is a better approximation to the exact solution (as obtained in [1]) than the classical RK4 method. Therefore, the DTM is a very efficient and accurate method that can be used to provide analytical solution for nonlinear differential equations.

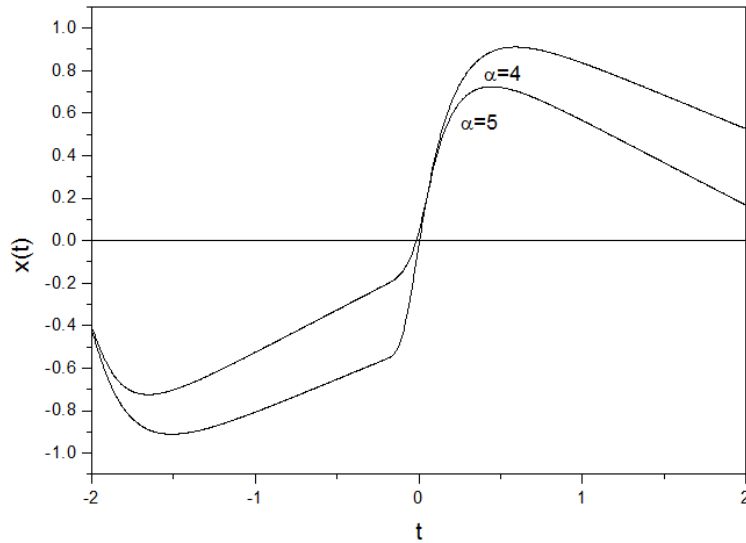


Fig. 1. Plot of solution (30) of Modified Equations of Emden type for the case $\alpha^2 = 8\beta$ taking $\alpha = 4$ and $\alpha = 5$

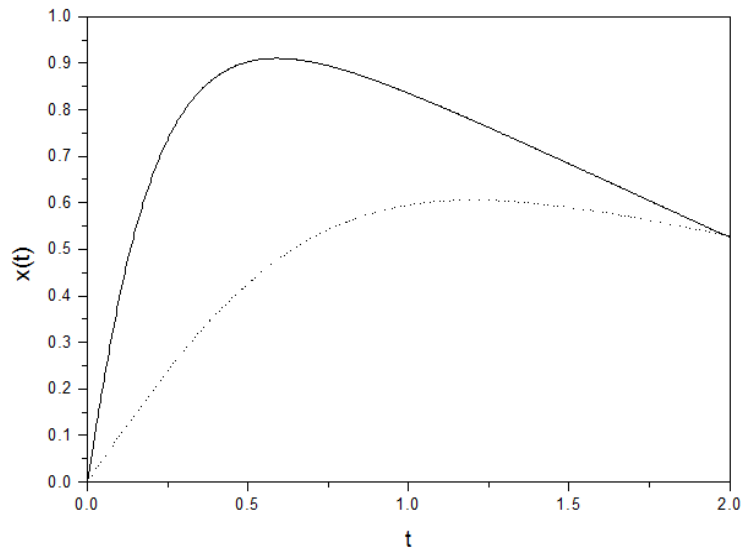


Fig. 2. Plot of solution of Modified Equations of Emden type for the case $\alpha^2 = 8\beta$ taking $\alpha = 4$ using DTM [Solid line] and RK4 method [Dotted line]

Table 1. Comparison of the DTM- solutions with RK4 solutions and calculation of Absolute error

Time	RK4 solution	DTM solution $x(t)$	Absolute error
0	0	0	0
0.1	0.09934	0.42503	0.32569
0.2	0.1948	0.67042	0.47562
0.3	0.28299	0.80679	0.5238
0.4	0.36135	0.87702	0.51567
0.5	0.42833	0.90712	0.47879
0.6	0.48334	0.91287	0.42953
0.7	0.52667	0.90384	0.37717
0.8	0.55915	0.88584	0.32669
0.9	0.58201	0.8624	0.28039
1	0.59662	0.83567	0.23905
1.1	0.60436	0.80693	0.20257
1.2	0.60652	0.77697	0.17045
1.3	0.60427	0.74628	0.14201
1.4	0.59858	0.71514	0.11656
1.5	0.5903	0.68373	0.09343
1.6	0.5801	0.65215	0.07205
1.7	0.56852	0.62048	0.05196
1.8	0.55602	0.58874	0.03272
1.9	0.54293	0.55697	0.01404
2	0.52952	0.52517	0.00435

6 Conclusion

The main concern of this paper was to construct an approximate analytical solution for the Modified Equations of Emden type. We have achieved this goal by applying DTM. The advantage of DTM is the fact that it provides its user with an analytical approximation. The results confirm that the proposed differential transform method is an effective means of solving various linear and nonlinear differential equations. They also indicate that a small number of arguments are sufficient to provide an accurate solution in practice. An implication of this is that, only a limited number of terms are required to be summed, and therefore the solution may be computed rapidly. However the method gives more satisfactory results for small times which are evident from Fig. 1 and we may get better results by adding more terms on the right hand side of equation (30) for longer time intervals.

Competing Interests

Authors have declared that no competing interests exist.

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A Discussion of Hausdorff Property on Fs-Cartesian Product Topological Spaces

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Abstract

For any nonempty family $\{\mathcal{B}_i, \mathcal{X}_i\}$ of FsB-Hausdorff Spaces. The Fs- Cartesian product topological space is also an FsB-Hausdorff Space.

Keywords: Fs-Set; Fs-Subset; (b, β) object; Fs-Point; FsB-Topological Space and FsB-Hausdorff Space.

1 Introduction

Ever since Zadeh [1] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets [2]. Nistla V.E.S. Murthy [3] introduced f-set in order to prove Axiom of choice for fuzzy sets which is not true for L-fuzzy sets [4]. Nistala V.E.S. Murthy [5] introduced the f-complement of an f-subset in [5]. We can easily see that the collection all f-subsets of a given f-set with this definition of f-complement could not form a Boolean algebra. Vaddiparthi Yogeswara, G. Srinivas and Biswajit Rath introduced the concept of Fs-sets and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-union, Fs- intersection and Fs-complements [6]. The Fs-sets they introduced contain Boolean valued membership functions. They are successful in their efforts in proving that result with some conditions [7]. Also Vaddiparthi Yogeswara, Biswajit Rath, Ch. RamaSanyaasi Rao, K. V. UmaKameswari, D. Raghu Ram introduced the concept of FsB-topological Space [8] on a given Fs-subset of an Fs-set and also they introduced FsB-subspace in the same paper. Fs-points and Fs-point set $FSP(\mathcal{A})$ are introduced by Vaddiparthi Yogeswara etc...[9] and based on Fs-set theory they defined a pair of relations between collection of all crisp subsets of Fs-points set $FSP(\mathcal{A})$ of the same Fs-set \mathcal{A} [10] and collection of all Fs-subsets of \mathcal{A} and proved one of the relations is a meet complete homomorphism and the other is a join complete homomorphism and searched properties of relations between Fs-complemented sets and complemented constructed crisp sets via these homomorphisms and ultimately they proved a representation theorem connecting Fs-subsets of \mathcal{A} to crisp subsets of $FSP(\mathcal{A})$ via homomorphisms. In this paper we introduce the concepts of T_1 - Space and Hausdorff Space on an Fs-B topological Space via these representation theorems and we give an example. In first four sections of this paper, we introduce Fs-sets, Fs-set functions etc ... in brief for smooth reading of the paper. We denote the largest element of a complete Boolean algebra L_A by M_A or 1_A . We denote Fs-union and crisp set union by the same symbol \cup and similarly Fs-intersection and crisp set intersection by the same symbol \cap etc... For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [11], Garret Birkhoff [12], Steven Givant • Paul Halmos [13] and Thomas Jech [14]. For results in topology one can refer [15].

2 Basic Definitions, Definitions and Theorems

2.1 Fs-set

Let U be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple $\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$ is said to be an Fs-set if, and only if

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- (1) $A \subseteq A_1$
- (2) L_A is a complete Boolean Algebra
- (3) $\mu_{1A_1}: A_1 \rightarrow L_A, \mu_{2A}: A \rightarrow L_A$, are functions such that $\mu_{1A_1}|A \geq \mu_{2A}$
- (4) $\bar{A}: A \rightarrow L_A$ is defined by $\bar{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c$, for each $x \in A$

2.2 Fs-subset

Let $\mathcal{A}=(A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$ and $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ be a pair of Fs-sets. \mathcal{B} is said to be an Fs-subset of \mathcal{A} , denoted by $\mathcal{B} \subseteq \mathcal{A}$, if, and only if

- (1) $B_1 \subseteq A_1, A \subseteq B$
- (2) L_B is a complete subalgebra of L_A or $L_B \leq L_A$
- (3) $\mu_{1B_1} \leq \mu_{1A_1}|B_1$, and $\mu_{2B}|A \geq \mu_{2A}$

2.3 Fs-union

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ be a pair of Fs-subsets of \mathcal{A} . Then, the Fs-union of \mathcal{B} and \mathcal{C} , denoted by $\mathcal{B} \cup \mathcal{C}$ is defined as

$\mathcal{B} \cup \mathcal{C} = \mathcal{D} = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$, where

- (1) $D_1 = B_1 \cup C_1, D = B \cap C$
- (2) $L_D = L_B \vee L_C =$ The complete subalgebra generated by $L_B \cup L_C$
- (3) $\mu_{1D_1}: D_1 \rightarrow L_D$ is defined by

$$\mu_{1D_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$$

$\mu_{2D}: D \rightarrow L_D$ is defined by

$$\mu_{2D}x = \mu_{2B}x \wedge \mu_{2C}x \text{ and}$$

$\bar{D}: D \rightarrow L_D$ is defined by

$$\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$$

2.4 Fs-intersection

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ be a pair of Fs-subsets of \mathcal{A} . Then, the Fs-intersection of \mathcal{B} and \mathcal{C} , denoted by $\mathcal{B} \cap \mathcal{C}$ is defined as

$\mathcal{B} \cap \mathcal{C} = \mathcal{D} = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$, where

- (1) $L_D = L_B \wedge L_C =$ The complete subalgebra generated by $L_B \cap L_C$
- (2) $\mu_{1D_1}: D_1 \rightarrow L_D$ is defined by

$$\mu_{1D_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$$

$\mu_{2D}: D \rightarrow L_D$ is defined by

$$\mu_{2D}x = (\mu_{2B} \vee \mu_{2C})x \text{ and}$$

$\bar{D}: D \rightarrow L_D$ is defined by

$$\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$$

2.5 FsB-TopologicalSpace

Suppose $\mu_{1A_1} = 1, \mu_{2A} = 0$ in \mathcal{A} . $\mathfrak{X} \subseteq \mathcal{L}(\mathcal{A})$ is said to be FsB-topology if, and only if

- 1) $(\mathcal{B}_i)_{i \in I} \subseteq \mathfrak{X} \Rightarrow \bigcup_{i \in I} \mathcal{B}_i \in \mathfrak{X}$
- 2) $(\mathcal{B}_i)_{i \in I}, I$ is finite set $\Rightarrow \bigcap_{i \in I} \mathcal{B}_i \in \mathfrak{X}$.

The pair $(\mathcal{A}, \mathfrak{T})$ is called an FsB-topological space.

Elements of \mathfrak{X} are called FsB-open ses or FsB-open subset of \mathcal{A} .

2.6 (b, β)- Object: Definition

Let $b \in A, \beta \in L_A$ such that $\beta \leq \bar{A}b$. we define a (b, β) -object, denoted by (b, β) itself as follows for $A \subseteq B \subseteq B_1 \subseteq A_1, L_B \leq L_A$, such that $\mu_{1B_1}x, \mu_{2B}x \in L_B(b, \beta) = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

$$\mu_{1B_1}x = \begin{cases} \mu_{2A}x, & x \neq b, x \in A \\ \beta \vee \mu_{2A}b, & x = b \\ \alpha, & x \notin A, x \in A_1 \end{cases} \quad \text{And } \mu_{2B}x = \begin{cases} \mu_{2A}x, & x \in A \\ \alpha, & x \notin A, x \in B \end{cases}$$

Here $\alpha \in L_A$ is fixed and $\alpha \leq \mu_{1A_1}x, \forall x \in A_1$

2.7 R(b, β)Relation

For any (b, β) objects $\mathcal{B}_1 = (B_{11}, B_1, \bar{B}_1(\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, \bar{B}_2(\mu_{1B_{12}}, \mu_{2B_2}), L_{B_2})$ of \mathcal{A} , we say that $\mathcal{B}_1 R(b, \beta) \mathcal{B}_2$ if, and only if $\mu_{1B_{11}}x = \mu_{2B_1}x, x \neq b$ and $\forall x \in B_1$ and $\mu_{1B_{12}}x = \mu_{2B_2}x, x \neq b$ and $\forall x \in B_2$ and $\mu_{1B_{11}}b = \mu_{1B_{12}}b = \beta \vee \mu_{2A}b$ and $\mu_{2B_1}b = \mu_{2B_2}b = \mu_{2A}b$.

We can easily show that $R(b, \beta)$ is an equivalence relation

2.8 Fs-point

The equivalence class corresponding to (b, β) is denoted by χ_b^β or (b, β) . We define this χ_b^β is an Fs-point of \mathcal{A} . Set of all Fs-point of \mathcal{A} is denoted by $FSP(\mathcal{A})$.

2.9 Definition

An Fs-B topological space $(\mathcal{B}, \mathfrak{X})$ is said to be T_1 -Space iff $\{\chi_b^\beta\}$ is closed for any $\chi_b^\beta \in \mathcal{A}^\sim$.

2.10 Definition

An Fs-B topological space $(\mathcal{B}, \mathfrak{X})$ is said to be T_2 -Space or FsB- Hausdroff space or simply Hausdroff space iff $\chi_{a_1}^{\alpha_1}, \chi_{a_2}^{\alpha_2} \in \mathcal{A}^\sim$ such that $\chi_{a_1}^{\alpha_1} \neq \chi_{a_2}^{\alpha_2}$ then there exists a pair of disjoint Fs-open sets \mathcal{G}_1 and \mathcal{G}_2 such that $\chi_{a_1}^{\alpha_1} \in \mathcal{G}_1^\sim, \chi_{a_2}^{\alpha_2} \in \mathcal{G}_2^\sim$.

3 Definitions and Theorems

3.1 Fs-cartesian product

Let $(\mathcal{A}_i)_{i \in I}$ be a non empty family of non empty Fs-sets.

Define Fs-Cartesian Product of $(\mathcal{A}_i)_{i \in I}$, denoted by $\prod_{i \in I} \mathcal{A}_i$ as follows.

Here $\mathcal{A}_i = (A_{1i}, A_i, \bar{A}_i(\mu_{1A_{1i}}, \mu_{2A_i}), L_{A_i})$, L_{A_i} is a non-degenerating complete Boolean algebra and $\bar{A}_i a_i \neq 0$ for at least one $a_i \in A_i$

Here $\prod_{i \in I} \mathcal{A}_i = \mathcal{X} = (X_1, X, \bar{X}(\mu_{1X_1}, \mu_{2X}), L_X)$, where

$X_1 = \prod_{i \in I} A_{1i}$ such that $(\prod_{i \in I} A_{1i}, (P_{1i})_{i \in I})$ is the product of $(A_{1i})_{i \in I}$ in SET, the category of sets with usual maps between crisp sets.

$X = \prod_{i \in I} A_i$ such that $(\prod_{i \in I} A_i, (P_i)_{i \in I})$ is the product of $(A_i)_{i \in I}$ in SET, the category of sets with usual maps between crisp sets.

$L_X = \prod_{i \in I} L_{A_i}$ such that $(\prod_{i \in I} L_{A_i}, (\pi_i)_{i \in I})$ is the product of $(L_{A_i})_{i \in I}$ in CBOO, the category of complete Boolean algebras with complete homomorphism between complete Boolean algebras.

$$\mu_{1X_1} = \prod_{i \in I} \mu_{1A_{1i}} : \prod_{i \in I} A_{1i} \rightarrow \prod_{i \in I} L_{A_i}$$

$$(a_i)_{i \in I} \mapsto (\mu_{1A_{1i}} P_{1i}(a_i)_{i \in I}) = (\mu_{1A_{1i}} a_i)_{i \in I}$$

$$\mu_{2X} = \prod_{i \in I} \mu_{2A_i} : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} L_{A_i}$$

$$(a_i)_{i \in I} \mapsto (\mu_{2A_i} P_i(a_i)_{i \in I}) = (\mu_{2A_i} a_i)_{i \in I}$$

$$\bar{X} = \prod_{i \in I} \bar{A}_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} L_{A_i}$$

$(a_i)_{i \in I} \mapsto (\bar{A}_i P_i(a_i)_{i \in I}) = (\bar{A}_i a_i)_{i \in I} = [\mu_{1A_{1i}} a_i \wedge (\mu_{2A_i} a_i)^c]_{i \in I}$ is an Fs-set

The Fs-function $(P_{1i}, P_i, \pi_i): \mathcal{X} \rightarrow \mathcal{A}_i$ are Fs-projections

In particular $\prod_{i \in I} \mathcal{A}_i = \mathcal{X} = \mathcal{A}^I$ where $\mathcal{A}_i = \mathcal{A}, \forall i \in I$

Theorem: $\prod_{i \in I} \mathcal{H}_i \cap \prod_{i \in I} \mathcal{K}_i = \prod_{i \in I} (\mathcal{H}_i \cap \mathcal{K}_i)$

3.2 Theorem

For each $i \in I, \mathcal{H}_i, \mathcal{K}_i \subseteq \mathcal{U}_i \prod_{i \in I} \mathcal{H}_i \cap \prod_{i \in I} \mathcal{K}_i = \prod_{i \in I} (\mathcal{H}_i \cap \mathcal{K}_i)$

Proof: $\prod_{i \in I} \mathcal{H}_i = \mathcal{U} = (U_1, U, \bar{U}(\mu_{1U_1}, \mu_{2U}), L_U)$, where

$$U_1 = \prod_{i \in I} H_{1i} \text{ such that } (\prod_{i \in I} H_{1i}, (P_{1i})_{i \in I}) \text{ is the product of } (H_{1i})_{i \in I}$$

$U = \prod_{i \in I} H_i$ such that $(\prod_{i \in I} H_i, (P_i)_{i \in I})$ is the product of $(H_i)_{i \in I}$

$L_U = \prod_{i \in I} L_{H_i}$ such that $(\prod_{i \in I} L_{H_i}, (\pi_i)_{i \in I})$ is the product of $(L_{H_i})_{i \in I}$

$\mu_{1U_1} = \prod_{i \in I} \mu_{1H_{1i}} : \prod_{i \in I} H_{1i} \rightarrow \prod_{i \in I} L_{H_i}$

$$(a_i)_{i \in I} \mapsto (\mu_{1H_{1i}} P_{1i}(a_i)_{i \in I}) = (\mu_{1H_{1i}} a_i)_{i \in I}$$

$\mu_{2U} = \prod_{i \in I} \mu_{2H_i} : \prod_{i \in I} H_i \rightarrow \prod_{i \in I} L_{H_i}$

$$(a_i)_{i \in I} \mapsto (\mu_{2H_i} P_i(a_i)_{i \in I}) = (\mu_{2H_i} a_i)_{i \in I}$$

$\bar{U} = \prod_{i \in I} \bar{H}_i : \prod_{i \in I} H_i \rightarrow \prod_{i \in I} L_{H_i}$

$$(a_i)_{i \in I} \mapsto (\bar{H}_i P_i(a_i)_{i \in I}) = (\bar{H}_i a_i)_{i \in I} = [\mu_{1H_{1i}} a_i \wedge (\mu_{2H_i} a_i)^c]_{i \in I}$$

$\prod_{i \in I} \mathcal{K}_i = \mathcal{V} = (V_1, V, \bar{V}(\mu_{1V_1}, \mu_{2V}), L_V)$, where

$V_1 = \prod_{i \in I} K_{1i}$ such that $(\prod_{i \in I} K_{1i}, (P_{1i})_{i \in I})$ is the product of $(K_{1i})_{i \in I}$

$V = \prod_{i \in I} K_i$ such that $(\prod_{i \in I} K_i, (P_i)_{i \in I})$ is the product of $(K_i)_{i \in I}$

$L_V = \prod_{i \in I} L_{K_i}$ such that $(\prod_{i \in I} L_{K_i}, (\pi_i)_{i \in I})$ is the product of $(L_{K_i})_{i \in I}$

$\mu_{1V_1} = \prod_{i \in I} \mu_{1K_{1i}} : \prod_{i \in I} K_{1i} \rightarrow \prod_{i \in I} L_{K_i}$

$$(a_i)_{i \in I} \mapsto (\mu_{1K_{1i}} P_{1i}(a_i)_{i \in I}) = (\mu_{1K_{1i}} a_i)_{i \in I}$$

$\mu_{2V} = \prod_{i \in I} \mu_{2K_i} : \prod_{i \in I} K_i \rightarrow \prod_{i \in I} L_{K_i}$

$$(a_i)_{i \in I} \mapsto (\mu_{2K_i} P_i(a_i)_{i \in I}) = (\mu_{2K_i} a_i)_{i \in I}$$

$\bar{V} = \prod_{i \in I} \bar{K}_i : \prod_{i \in I} K_i \rightarrow \prod_{i \in I} L_{K_i}$

$$(a_i)_{i \in I} \mapsto (\bar{K}_i P_i(a_i)_{i \in I}) = (\bar{K}_i a_i)_{i \in I} = [\mu_{1K_{1i}} a_i \wedge (\mu_{2K_i} a_i)^c]_{i \in I}$$

L.H.S = $\mathcal{U} \cap \mathcal{V} = \mathcal{Y} = (Y_1, Y, \bar{Y}(\mu_{1Y_1}, \mu_{2Y}), L_Y)$ where

$Y_1 = U_1 \cap V_1 = \prod_{i \in I} H_{1i} \cap \prod_{i \in I} K_{1i} = \prod_{i \in I} (H_{1i} \cap K_{1i})$

$Y = U \cup V = (\prod_{i \in I} H_i) \cup (\prod_{i \in I} K_i) = \prod_{i \in I} (H_i \cap K_i)$

$L_Y = L_U \cap L_V$

$\mu_{1Y_1} = \mu_{1U_1} \wedge \mu_{1V_1} = (\mu_{1H_{1i}})_{i \in I} \wedge (\mu_{1K_{1i}})_{i \in I}$

$= (\mu_{1H_{1i}} \wedge \mu_{1K_{1i}})_{i \in I}$

$\mu_{2Y} = \mu_{2U} \vee \mu_{2V} = (\mu_{2H_i})_{i \in I} \vee (\mu_{2K_i})_{i \in I}$

$= (\mu_{2H_i} \vee \mu_{2K_i})_{i \in I}$

$$\mathcal{H}_i \cap \mathcal{K}_i = \mathcal{W}_i$$

$$\mathcal{W}_i = (W_{1i}, W_i, \overline{W}_i(\mu_{1W_{1i}}, \mu_{2W_i}), L_{W_i})$$

$$\prod_{i \in I} \mathcal{W}_i = \mathcal{X} = (X_1, X, \overline{X}(\mu_{1X_1}, \mu_{2X}), L_X), \text{ where}$$

$$X_1 = \prod_{i \in I} W_{1i} \text{ such that } (\prod_{i \in I} W_{1i}, (P_{1i})_{i \in I}) \text{ is the product of } (W_{1i})_{i \in I}$$

$$X = \prod_{i \in I} W_i \text{ such that } (\prod_{i \in I} W_i, (P_i)_{i \in I}) \text{ is the product of } (W_i)_{i \in I}$$

$$L_X = \prod_{i \in I} L_{W_i} \text{ such that } (\prod_{i \in I} L_{W_i}, (\pi_i)_{i \in I}) \text{ is the product of } (L_{W_i})_{i \in I}$$

$$\mu_{1X_1} = \prod_{i \in I} \mu_{1W_{1i}} : \prod_{i \in I} W_{1i} \rightarrow \prod_{i \in I} L_{W_i}$$

$$(a_i)_{i \in I} \mapsto (\mu_{1W_{1i}} P_{1i}(a_i)_{i \in I}) = (\mu_{1W_{1i}} a_i)_{i \in I}$$

$$\mu_{2X} = \prod_{i \in I} \mu_{2W_i} : \prod_{i \in I} W_i \rightarrow \prod_{i \in I} L_{W_i}$$

$$(a_i)_{i \in I} \mapsto (\mu_{2W_i} P_i(a_i)_{i \in I}) = (\mu_{2W_i} a_i)_{i \in I}$$

$$\overline{X} = \prod_{i \in I} \overline{W}_i : \prod_{i \in I} W_i \rightarrow \prod_{i \in I} L_{W_i}$$

$$(a_i)_{i \in I} \mapsto (\overline{W}_i P_i(a_i)_{i \in I}) = (\overline{W}_i a_i)_{i \in I} = [\mu_{1W_{1i}} a_i \wedge (\mu_{2W_i} a_i)]_{i \in I}^c$$

$$\text{So we can proved that } \prod_{i \in I} \mathcal{H}_i \cap \prod_{i \in I} \mathcal{K}_i = \prod_{i \in I} (\mathcal{H}_i \cap \mathcal{K}_i)$$

Let $\{(\mathcal{B}_i, \mathfrak{A}_i)\}$ Housdorff space

$$\prod_{i \in I} \mathcal{B}_i = \mathcal{C} = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C), \text{ where}$$

$$C_1 = \prod_{i \in I} B_{1i} \text{ such that } (\prod_{i \in I} B_{1i}, (P_{1i})_{i \in I}) \text{ is the product of } (B_{1i})_{i \in I}$$

$$C = \prod_{i \in I} B_i \text{ such that } (\prod_{i \in I} B_i, (P_i)_{i \in I}) \text{ is the product of } (B_i)_{i \in I}$$

$$L_C = \prod_{i \in I} L_{B_i} \text{ such that } (\prod_{i \in I} L_{B_i}, (\pi_i)_{i \in I}) \text{ is the product of } (L_{B_i})_{i \in I}$$

$$\mu_{1C_1} = \prod_{i \in I} \mu_{1B_{1i}} : \prod_{i \in I} B_{1i} \rightarrow \prod_{i \in I} L_{B_i}$$

$$(a_i)_{i \in I} \mapsto (\mu_{1B_{1i}} P_{1i}(a_i)_{i \in I}) = (\mu_{1B_{1i}} a_i)_{i \in I}$$

$$\mu_{2C} = \prod_{i \in I} \mu_{2B_i} : \prod_{i \in I} B_i \rightarrow \prod_{i \in I} L_{B_i}$$

$$(a_i)_{i \in I} \mapsto (\mu_{2B_i} P_i(a_i)_{i \in I}) = (\mu_{2B_i} a_i)_{i \in I}$$

$$\overline{C} = \prod_{i \in I} \overline{B}_i : \prod_{i \in I} B_i \rightarrow \prod_{i \in I} L_{B_i}$$

$$(a_i)_{i \in I} \mapsto (\overline{B}_i P_i(a_i)_{i \in I}) = (\overline{B}_i a_i)_{i \in I} = [\mu_{1B_{1i}} a_i \wedge (\mu_{2B_i} a_i)]_{i \in I}^c = (\overline{B}_i)_{i \in I}$$

3.3 Theorem

For each $i \in I$, $(\prod_{i \in I} \mathcal{B}_i)^\sim = \prod_{i \in I} \mathcal{B}_i^\sim$

$$\begin{aligned} \chi_a^\alpha \in \text{L.H.S} &\Rightarrow \mathcal{C}^\sim = \{ \chi_c^\gamma / c \in C, \gamma \in L_c, \gamma \leq \bar{C}c \} \\ \{ \chi_c^\gamma / c &= (C_i)_{i \in I} \in \prod_{i \in I} B_i, \gamma = (\gamma_i)_{i \in I} \in \prod_{i \in I} L_{B_i}, \\ \gamma &= (\gamma_i)_{i \in I} \leq \bar{C}c = (\bar{B}_i C_i)_{i \in I} \} \\ \Leftrightarrow \chi_a^\alpha &\text{ where } a = (a_i)_{i \in I} \in \prod_{i \in I} B_i, \alpha = (\alpha_i)_{i \in I} \in \prod_{i \in I} L_{B_i}, \alpha = (\alpha_i)_{i \in I} \leq \bar{C}a \\ &= (\bar{B}_i a_i)_{i \in I} \\ \Leftrightarrow a_i \in B_i, \alpha_i &\in L_{B_i}, \alpha_i \leq \bar{B}_i a_i \\ \Leftrightarrow \chi_{a_i}^{\alpha_i} \in \mathcal{B}_i &\Rightarrow (\chi_{a_i}^{\alpha_i})_{i \in I} \in \prod_{i \in I} \mathcal{B}_i^\sim \end{aligned}$$

3.4 Definition

Define $\chi_a^\alpha = (\chi_{a_i}^{\alpha_i})_{i \in I}$ where $a = (a_i)_{i \in I}, \alpha = (\alpha_i)_{i \in I}$

3.5 Theorem

To prove Hausdorff Property of product space

$$\begin{aligned} \text{Let } \chi_a^\alpha, \chi_b^\beta &\in (\prod_{i \in I} \mathcal{B}_i)^\sim = \prod_{i \in I} \mathcal{B}_i^\sim \text{ and } \chi_a^\alpha \neq \chi_b^\beta \\ \Rightarrow \chi_{a_{i_0}}^{\alpha_{i_0}} &\neq \chi_{b_{i_0}}^{\beta_{i_0}} \text{ for at least one } i_0 \in I \end{aligned}$$

There exist a pair of disjoint Fs – open sets $\mathcal{G}_{i_01}, \mathcal{G}_{i_02}$ in \mathcal{B}_{i_0} there exist

$$\mathcal{G}_{i_01} \cap \mathcal{G}_{i_02} = \Phi_A \text{ and } \chi_{a_{i_0}}^{\alpha_{i_0}} \in \mathcal{G}_{i_01}^\sim, \chi_{b_{i_0}}^{\beta_{i_0}} \in \mathcal{G}_{i_02}^\sim$$

$$\text{Defined } \mathcal{P} = \prod_{i \in I} \mathcal{P}_i, \mathcal{P}_i = \mathcal{B}_i \forall i \neq i_0, \mathcal{P}_{i_0} = \mathcal{G}_{i_01},$$

$$\mathcal{Q} = \prod_{i \in I} \mathcal{Q}_i, \mathcal{Q}_i = \mathcal{B}_i \forall i \neq i_0, \mathcal{Q}_{i_0} = \mathcal{G}_{i_02}$$

$$\begin{aligned} \text{Here } \mathcal{P} \cap \mathcal{Q} &= \prod_{i \in I} \mathcal{P}_i \cap \prod_{i \in I} \mathcal{Q}_i \\ &= \prod_{i \in I} (\mathcal{P}_i \cap \mathcal{Q}_i) \\ &= \prod_{i \neq i_0} (\mathcal{P}_i \cap \mathcal{Q}_i) (\mathcal{P}_{i_0} \cap \mathcal{Q}_{i_0}) \\ &= \Phi_A \end{aligned}$$

$$\text{And observe that } \chi_a^\alpha \in \mathcal{P} = \prod_{i \in I} \mathcal{P}_i$$

$$\chi_b^\beta \in \mathcal{Q} = \prod_{i \in I} \mathcal{Q}_i$$

Here P, Q are defining Fs – sub basic open sets in Product topology.

4 Conclusion

The Hausdorff property, which is very important concept in general topological spaces is introduced in the theory of FsB-topological spaces also.

Competing Interests

Authors have declared that no competing interests exist.

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Arrivals Analysis

Nirmala Kasturi^{1*}

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Abstract

We need many analyses to come to an inference about any situation. Probability Distributions play very important role in such analyses. We propose one Probability Distribution function of random variable successive arrivals.

Keywords: Random variable; continuous probability distribution; arrival rate; density function.

1 Introduction

We have seen the random variables “Time for the first arrival”, “time for the r^{th} arrival”, etc. Now the random variable of interest is instead of asking “In fixed time interval, how many departures take place?”, “We ask in a particular interval how likely there are successive arrivals” [1-25]. This random variable is continuous if it holds

up a p.d.f. $f(x)$ which is continuous over the time axis with $f(x)$ greater than or equal to zero and $\int_{-\infty}^{\infty} f(x)dx = 1$

‘0’ number of successive arrivals will be in the system not only at starting time, system may idle for some time when entered arrivals have taken the service and left the system [26].

The following Fig. 1 is the histogram of the data when the system is with 0 no. of successive arrivals in different time periods.

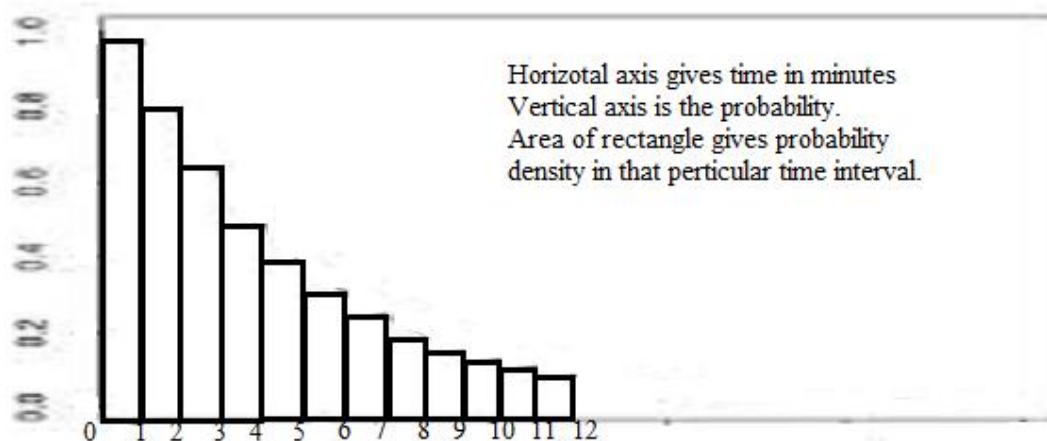


Fig. 1. Histogram - 1

The following Fig. 2 is the density function obtained from the histogram of data when the system is with 0 no. of successive arrivals in different time periods.

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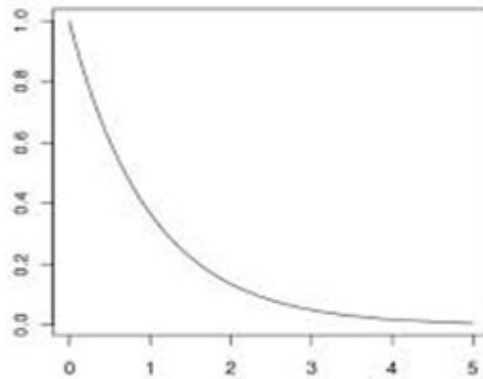


Fig. 2. Density curve - 1
X - represents time, Y - represents $f(x)$

The following Fig. 3 is the density function obtained from the histogram of data when the system is with 1 no. of successive arrivals in different time periods [26].

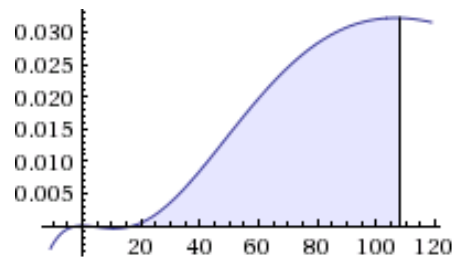


Fig. 3. Density curve - 2
X - represents time, Y - represents $f(x)$

The following Fig. 4 is the density function obtained from the histogram of data when the system is with 2 no. of successive arrivals in different time periods [26].

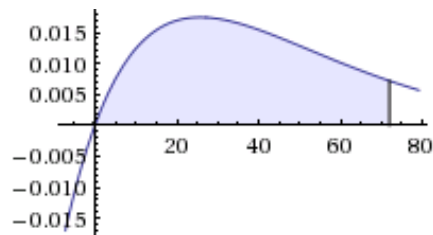


Fig. 4. Density curve - 3
X - represents time, Y - represents $f(x)$

By drawing the histograms and respective density curves along the top of the rectangle bars of histograms we calculated the probability Distribution function by using regression techniques. Here we propose [27].

$$f(x) = \begin{cases} \lambda [2\Im(P(n, x) - P(n-1, x))] & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for corresponding random variable where $P(n, t)$ is probability density function (Poisson) [28].

Verification: [26]

For $n = 0$ } Cases in the above equation are absurd since we consider the two consecutive arrivals in the
 $n = 1$ } time interval.

We consider $n = 2$.

Case 1: $n = 2$

Here we propose

$$f(x) = \lambda [2p(2,t) - P(1,t)]$$

$$f(x) = \lambda \left[2 \frac{e^{-\lambda t} (\lambda t)^2}{2!} - e^{-\lambda t} \lambda t \right]$$

Here $f(x)$ is integrable and in general probability of arriving one customer in particular interval is greater than probability of arriving customers in same time interval.

Also twice the $P(2, t)$ is more than $P(1, t)$

$[2P(2, t) - P(1, t)]$ will give absolutely a positive number.

The arrival rate λ is positive quantity.

$$\therefore \lambda [2P(2,x) - P(1,x)] \geq 0$$

$$\therefore f(x) \geq 0 \forall x \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx$$

Now $\int_0^{\infty} \lambda [2P(2,t) - P(1,t)] dt$

$$\int_0^{\infty} \lambda \left[2 \frac{(\lambda t)^2 e^{-\lambda t}}{2!} - \frac{e^{-\lambda t} (\lambda t)^1}{1!} \right] dt$$

$$\int_0^{\infty} \left[2 \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \lambda - \frac{e^{-\lambda t} (\lambda t)^1}{1!} \lambda \right] dt$$

$$2(1) - 1 = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx = 1 \text{ for } n = 2$$

Therefore the p.d.f is $f(x) = \lambda [2P(2,x) - P(1,x)]$.

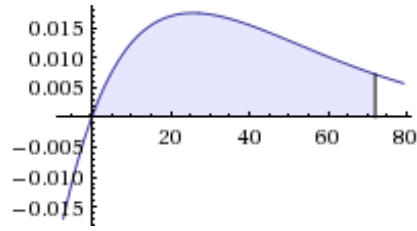


Fig. 5. Density curve - 4
X - represents time, Y - represents $f(x)$

Case – 2: $n = 3$

$$f(x) = \lambda [2P(3, t) - P(2, t)]$$

$$= \lambda \left[2 \frac{e^{-\lambda x} (\lambda x)^3}{3!} - \frac{e^{-\lambda x} (\lambda x)^2}{2!} \right]$$

Here $f(x)$ is integrable and in general probability of arriving two customers in particular interval is greater than probability of arriving three customers in that time interval.

Also twice the $P(3, t)$ is more than $P(2, t)$.

$2P(3, t) - P(2, t)$ is a positive quantity.

The arrival rate λ is a positive value.

$$\therefore f(x) \geq 0 \forall x$$

$$\therefore \lambda [2P(3, x) - P(2, x)] \geq 0$$

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx$$

$$\int_0^{\infty} \lambda [2P(3, t) - P(2, t)] dt$$

$$\int_0^{\infty} \lambda \left[2 \frac{(\lambda t)^3 e^{-\lambda t}}{3!} - \frac{e^{-\lambda t} (\lambda t)^2}{2!} \right] dt$$

$$\int_0^{\infty} \left[2 \frac{(\lambda t)^3 e^{-\lambda t}}{3!} \lambda - \frac{e^{-\lambda t} (\lambda t)^2}{2!} \lambda \right] dt$$

$$2(1) - 1 = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx = 1 \text{ for } n = 3$$

$f(x) = \lambda [2P(3, x) - P(2, x)]$ is p.d.f. function when $n = 3$.

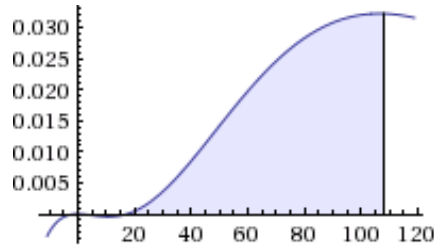


Fig. 6. Density curve - 5
X - represents time, *Y* - represents $f(x)$

Case - 3: $n = 4$

$$f(x) = \lambda [2P(4, x) - P(3, x)]$$

$$= \lambda \left[2 \frac{e^{-\lambda x} (\lambda x)^4}{4!} - \frac{e^{-\lambda x} (\lambda x)^3}{3!} \right]$$

Here $f(x)$ is integrable and since the probability of arriving four persons to system, $P(4, t)$ is less than probability of arriving 3 persons $P(3, t)$ to that system in the same time 't'.

Also twice $P(4, t)$ is more than $P(3, t)$

$2P(4, t) - P(3, t)$ is absolutely positive value.

The arrival rate λ is positive quantity.

$$\therefore \lambda [2P(4, t) - P(3, t)] \geq 0$$

$$\therefore f(x) \geq 0 \forall x \quad \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx$$

$$\int_0^{\infty} \lambda [2P(4, t) - P(3, t)] dt$$

$$\int_0^{\infty} \lambda \left[2 \frac{(\lambda t)^3 e^{-\lambda t}}{3!} - \frac{e^{-\lambda t} (\lambda t)^2}{2!} \right] dt$$

$$\int_0^{\infty} \left[2 \frac{(\lambda t)^4 e^{-\lambda t}}{4!} \lambda - \frac{e^{-\lambda t} (\lambda t)^3}{3!} \lambda \right] dt$$

$$2(1) - 1 = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx = 1 \text{ for } n = 4$$

$f(x) = \lambda [2P(4, x) - P(3, x)]$ is p.d.f. function when $n = 4$.

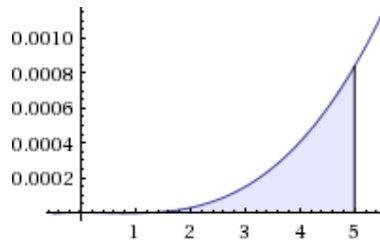


Fig. 7. Density curve - 6
 X - represents time, Y - represents $f(x)$

When it comes to the finite interval, we need to multiply the normalizer to get area under the curve equals to 1.

In general, we define $f(x) = \begin{cases} \lambda [2\mathfrak{S}(P(n, x) - P(n-1, x))] & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Where \mathfrak{S} is the normalizing constant [26].

2 Conclusion

For the random variable for the successive arrivals, the probability density function,

$$f(x) = \begin{cases} \lambda [2\mathfrak{S}(P(n, x) - P(n-1, x))] & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This works for all queuing systems in Poisson process [29].

3 Future Work

We can find CDF, standard deviation, mean and variance of the above distribution.

Competing Interests

Author has declared that no competing interests exist.

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Departures Analysis

Nirmala Kasturi^{1*}

DOI: 10.9734/bpi/rsmcs/v2

Abstract

To provide general expression towards the any statistical situation we need many models and also analyses to come to an inference. Probability Distribution plays very important role in such analyses. Among all statistical empirical distributions, several distributions have been developed by some subtle transformations on the existing distributions. We propose one Probability Distribution function of random variable successive departures.

Keywords: Random variable; continuous probability distribution; departure rate; density function.

1 Introduction

The random variable of interest is instead of asking “In fixed time interval, how many departures take place?”, “We ask in a particular interval how likely there are successive departures”. We can say this random variable is continuous, if there exist truncated probability density $f(x)$ which is continuous over the time axis with $f(x)$ is greater than or equal to zero for all x

$$\text{also } \int_{-\infty}^{\infty} f(x)dx = 1$$

Here we restrict the number of arrivals (domain) to the queuing system. We allow N initially [1-26]. Thus we consider truncated density function. The no. of persons remained later departing $N-n$ persons from the system after the service [27].

The following Fig. 1 is the histogram of the data when the system is with 0 no. of successive departures in different time periods.

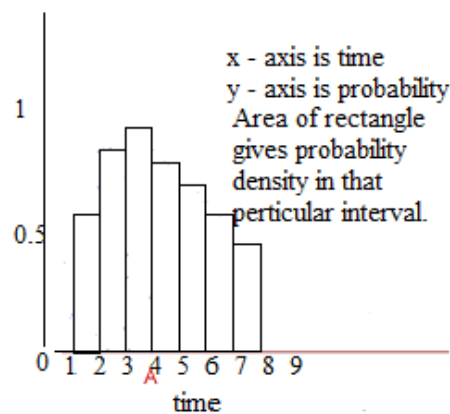


Fig. 1. Histogram - 1

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*Corresponding author: E-mail: vaka.nirmalaprakash@gmail.com;

The following Fig. 2 is the density function obtained from the histogram of data when the system is with 0 no. of successive departures in different time periods [27].

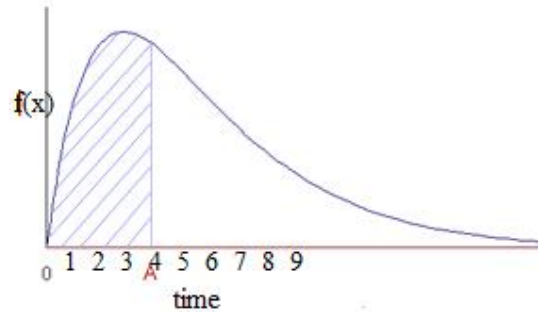


Fig. 2. Density function curve - 1

The following Fig. 3 is the density function obtained from the histogram of data when the system is with 1 no. of successive departures in different time periods [27].

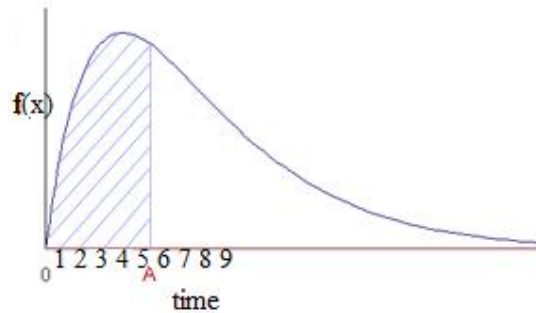


Fig. 3. Density function curve - 2

The following Fig. 4 is the density function obtained from the histogram of data when the system is with 2 no. of successive departures in different time periods [27].

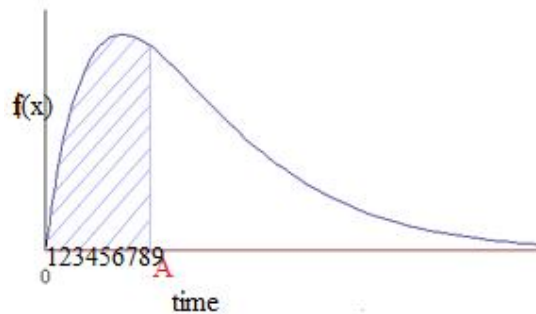


Fig. 4. Density function curve - 3

By drawing the histograms and respective density curves along the top of the rectangle bars of histograms we calculated the probability Distribution function by using regression techniques. Here we propose

$$f(x) = \begin{cases} 2\mathfrak{S}'P(n, x) - P(n - 1, x) & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for corresponding random variable where $P(n, t)$ is probability density function (Truncated Poisson) [28].

Verification: [27]

For $n = 0$ means $N - N = 0$ }
 $n = 1$ means $N - (N - 1) = 1$ } Cases are absurd since we consider the two consecutive departures in the time interval.

Case a: When n is 1: that is $n = N - 2$

We propose $f(x) = \mu [2P(N - (N - 2), t) - P(N - (N - 1), t)]$
 $f(x) = [\mu (2P(2, t) - P(1, t))]$

Where $P(1, t)$, $P(2, t)$ are Truncated Poisson probability distributions $P(1, t) = e^{-\mu t} \mu t$

$$P(2, t) = \frac{e^{-\mu t} (\mu t)^2}{2!} \quad [29]$$

Since probability of departing 2 persons from a queuing system $P(2, t)$ in time 't' is less than probability of departing 1 person $P(1, t)$ from that system in same time 't'.

Also twice the $P(2, t)$ is more than $P(1, t)$.

$\therefore (2P(2, t) - P(1, t))$ is a positive quantity.

μ is departing rate of the queuing system which is positive

$$\therefore f(x) = \mu [2P(2, t) - P(1, t)] \geq 0$$

$$\text{Now } \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx = 2 \int_0^{\infty} \mu P(2, t) dt - \int_0^{\infty} \mu P(1, t) dt$$

$$= 2 - 1 = 1$$

$\therefore f(x)$ is density function when n is 2.

Case b: When n is 3: n is no. of customers remained in the system later departing $N - 3$ persons We propose

$$f(x) = \mu [2P(N - (N - 3), t) - P(N - (N - 2), t)]$$

$$f(x) = \mu [2P(3, t) - P(2, t)]$$

$$P(3, t) = \frac{e^{-\mu t} (\mu t)^3}{3!}$$

$$P(2, t) = \frac{e^{-\mu t} (\mu t)^2}{2!}$$

μ is departing rate of the queuing system which is also positive.

Since the probability of departing 3 persons from a queuing system $P(3,t)$ in time 't' is less than probability of departing 2 persons $P(2, t)$ from that system in same time 't' .

Also twice the $P(3, t)$ is more than $P(2, t)$

$\therefore (2P(3,t) - P(2,t))$ is a positive quantity.

$$\therefore f(x) = \mu [2P(3,t) - P(2,t)] \geq 0$$

$$\text{Now } \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx = 2 \int_0^{\infty} \mu P(3,t) dt - \int_0^{\infty} \mu P(2,t) dt$$

$$= 2 - 1 = 1$$

$\therefore f(x)$ is density function when n is 3.

Case c : When n is 4

n is the no. of customers left in the system later departing $N - 4$ persons. We propose

$$f(x) = \mu [2P(N - (N - 4), t) - P(N - (N - 3), t)]$$

$$f(x) = \mu [2P(4,t) - P(3,t)]$$

The truncated Poisson probability distributions

$$P(4,t) = \frac{e^{-\mu t} (\mu t)^4}{4!}$$

$$P(3,t) = \frac{e^{-\mu t} (\mu t)^3}{3!}$$

μ is departure rate of the queuing system which is positive.

Since the probability of departing 4 persons from a queuing system $P(4,t)$ in time 't' is less than probability of departing 3 persons $P(3, t)$ from that system in same time 't' .

Also twice $P(4, t)$ is more than $P(3, t)$

$\therefore (2P(4,t) - P(3,t))$ is a positive quantity.

$$\therefore f(x) = \mu [2P(4,t) - P(3,t)] \geq 0$$

$$\text{Now } \int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx = 2 \int_0^{\infty} \mu P(4,t) dt - \int_0^{\infty} \mu P(3,t) dt = 2 - 1 = 1 [27]$$

Graphs of the density functions:

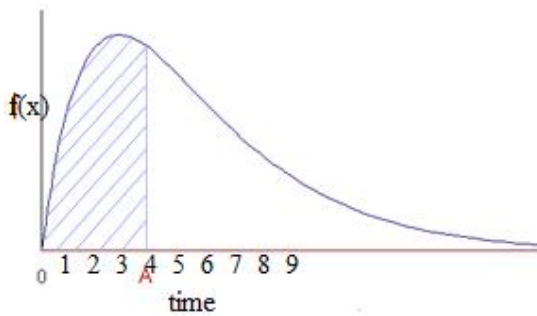


Fig. 5. Density function curve - 4

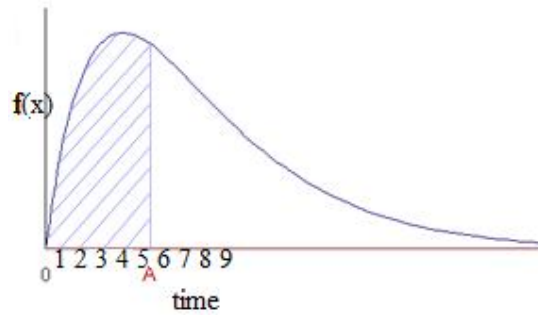


Fig. 6. Density function curve - 5

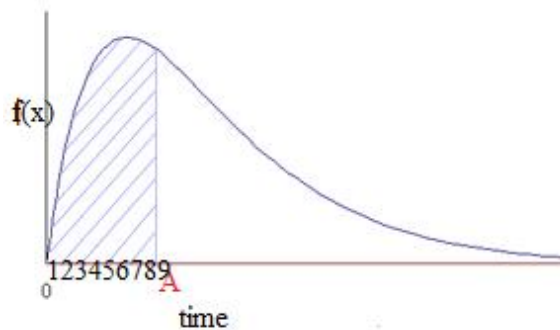


Fig. 7. Density function curve - 1
X axis - time and Y axis - f(x)

Shaded region is probability of consecutive departures in the interval $[0, A]$.

The probability density curve is not symmetric. When it comes to generalization, we need to normalize to keep the area under the curve equals to one. In general we consider for the random variable of the successive departures we specify the p.d.f. below.

$$f(x) = \begin{cases} 2\mathfrak{S}'P(n, x) - P(n-1, x) & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where \mathfrak{S}' is the normalizing constant [27]

The probability of consecutive departures increases when we increase the time interval.

2 Conclusion

In this we presented for the random variable of the successive departures, the Probability distribution function,

$$f(x) = \begin{cases} 2\mathfrak{S}'P(n, x) - P(n-1, x) & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This works for all queuing systems in Poisson process.

3 Future Work

We can find CDF, standard deviation, mean and variance of the above distribution.

Competing Interests

Author has declared that no competing interests exist.

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An Approach of Short Term Road Traffic Flow Forecasting Using Artificial Neural Network

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Abstract

In recent days, road traffic management and congestion control has become major problems in any busy junction in Hyderabad city. Hence short term traffic flow forecasting has gained greater importance in Intelligent Transport System (ITS). Artificial Neural Network (ANN) models have been fruitfully applied for classification and prediction of time series. In this chapter, an attempt has been made to model and forecast short-term traffic flow at 6.no. junction in Amberpet, Hyderabad, Telangana state, India applying Neural Network models. The traffic data has been considered for peak hours in the morning for 8A.M to 12 Noon, for 5 days. Multilayer Perceptron (MLP) network model is used in this study. These results can be considered to monitor traffic signals and explore methods to avoid congestion at that junction.

Keywords: Traffic volume; multilayer perceptron; artificial neural network; intelligent transport system; forecasting.

1 Introduction

India is the most populated country in the world and one of the fastest developed country. The rapid growth of personal vehicles (cars and two wheelers) in addition to private and public transport (cabs, tucks and RTC buses etc..) result in huge traffic congestion in most of the cities in India. In recent years Intelligent Transport System (ITS) is being applied to reduce congestion [1]. The prediction of short term traffic and future traffic conditions based on present and past traffic is a major component of Intelligent Transport System(ITS) applications. The importance of traffic flow forecasting for ITS has important applications such as development of traffic control strategies in Advanced Traffic Management Systems [2] and Advanced Traveler Information Systems [3].

Short term traffic flow forecasting involves predicting the traffic volume in the next time interval usually in the range of 5 minutes to 30 minutes. For this study we have considered 5days traffic data at 6 no. junction, Amberpet, Hyderabad, Telangana state, India. In any junction it is very important to forecast the short term traffic flow to design planning and operations of traffic signals and various traffic strategies. In this paper an attempt was made to develop a short term traffic flow forecasting model using Artificial Neural Network(ANN).

The field of Neural Network was established before the invention of computers, but Neural Network imitations are contemporary evolution. In 1943 the elemental artificial neuron was invented by Warren Mc Culloch, neurophysiologist. After that artificial neurons were proposed by Hebbian (1949) and Perceptron (1958). The single layer networks which are called perceptrons were introduced by Frank Rosenblatt.

Neural Network (NN) solves problem by different approach than that of conventional computers. Neural Network exercise instruction alike as the human brain exercises. The NN is made of an immense amount of inter linked neurons which are the processing elements they work parallel in solving a particular problem. They are well suited for structure identification and categorization [4]. ANNs got greater prominence in analysis of time series as well as forecasting [5]. The ANN aims to pursue to identify structures in a given set. After the NN trained on given set, it detects similar pattern in future data to make predictions (Mohie El-Din et al., 2017)

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An Artificial Neural Network(ANN) is an information processing paradigm that is inspired by the way biological nervous systems, such as brain, spinal cord process information. In this study MultiLayer Perceptron(MLP) network has been used for the prediction of short term traffic flow.

2 Materials and Methods

2.1 Data collection and processing

The data was taken from the office of Commissionerate, traffic, Hyderabad. The data set used in this study was collected from 6 no. junction, Amberpet, Hyderabad, Telangana state, India. Data was collected during 8:00 am to 12:00 noon from 08-02-2019 to 12-02-2019. The data was given in the form of video captured by cc TV cameras fixed at the junction. Volume of each category was obtained by counting the vehicles manually. Data extraction was done in the intervals of 5 minutes for 4 directions. The data consists of number of vehicles passing in the junction in all directions. The vehicles has been divided into 3 categories; 1.Two wheeler vehicles (Bike, Scooter, etc.) , 2. Three and Four wheeler vehicles (Car, Auto, Jeep, Mini truck etc..) 3. Heavy vehicles (Bus, lorry, Truck, Tractor etc..). The observed data was collected for the peak period of 4 hours for 5 days gave, a sample of 240 data points. The basic Statistical characteristics of three categories of vehicles are presented in Table 1.

Table 1. Statistical characteristics of traffic volume

Statistic	Two wheeler	3 and 4 wheelers	Heavy vehicles
Minimum	240	140	14
Maximum	492	424	48
Mean	357.654	257.342	21.95
Standard deviation	59.196	55.396	6.358

In this study three input variables were considered (the frequency of each category of vehicles as one variable) to create a neural network model to forecast the future traffic. 80% of the data set i.e; 192 data points were considered for training and the remaining 20% (48 data points) were considered for testing.

2.2 Methodology

Neural networks have broad applicability to real world. Since, neural networks are best for identifying patterns or trends in data, they are well suited for prediction or forecasting needs.

Feed-Forwarded Artificial Neural Networks enable signals to proceed from inserted data to result. In this the result of some layer will not alter the similar layer i.e. there is no feedback (loops). Feed-Forwarded Networks are uncomplicated architectures, they relate inserted data to result. Feed-Forward Networks are most-widely employed for structure identification. This organization is as well termed as top-down or bottom-up.

Activation function (AF) is applied to the weighted sum of the inputs of a neuron to give the output. The activation functions that are commonly employed are:

- Identify function: $f(x) = x$
- Binary step function: $f(x) \begin{cases} = 1 & \text{when } x < \theta \\ = 0 & \text{when } x \geq \theta \end{cases}$
- Sigmoid function: $\frac{1}{1+e^{-x}}$
- Bipolar sigmoid function: $(1 - e^{-x})(1 + e^{-x})^{-1}$

Two types of training algorithms used to train the Neural Networks.

In supervised learning, it checks at every step, the input that is applied with the desired output supplied by the model. It takes less no of iterations, less time and it produces more accuracy. Some of supervised learning are perceptron, Delta learning law and Back Propagation etc.

UN supervised learning, when the input is applied at every stage again it passes through the network. The process stops when the two adjacent outputs are same. It takes more number of iterations, more time and gives less accuracy. Examples are Hebbian learning, Competitive Learning and self organizing maps etc.

Multilayer perceptron(MLP) is most widely used network structure of Artificial Neural Network (ANN). A Multi-Layer perceptron is a class of feed forward Artificial Neural Network. Multilayer perceptron(MLP) is able to solve non linearly separable problems, a number of neurons connected in layers to build a Multilayer perceptron. Each of the perceptrons is used to identify small linearly separable sections of the inputs. Outputs of the perceptrons are combined into another perceptron to produce the final output [6]. The architecture of the MultiLayer Perceptron includes the neurons are arranged into an input layer an output layer and one or more hidden layers.

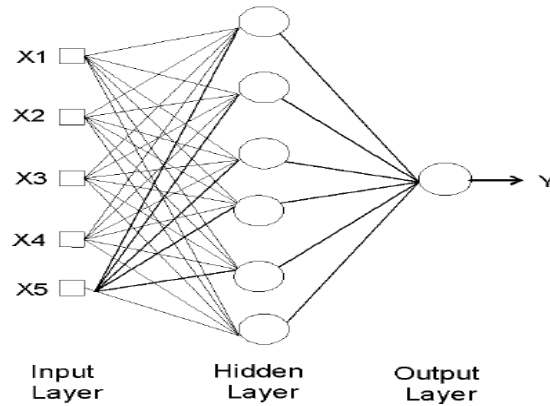


Fig. 1. Picture of multi layer perceptron network

MultiLayer Perceptron uses the “back propagation rule” which calculates an error function for each input and back propagates the error from one layer to the previous one. The weights for a particular node are adjusted indirect proportion to the error in the units to which it is connected. An activation function is applied to the weighted sum of the inputs of a neuron to produce the output. In this study we used sigmoid function as activation function.

Sigmoid function: $\frac{1}{1+e^{-x}}$

The MLP learning algorithm using the back propagation rule includes initialise weights (to small random values) and transfer function and adjust weights by starting from output layer and working backwards.

$$W_{ij}(t + 1) = W_{ij}(t) + \eta \delta_{pj} O_{pj}$$

Where $W_{ij}(t)$ represents the weights from node i to node j at time t ,

η is a gain term and δ_{pj} is an error term for pattern p on node j (where the sum is over the k nodes in the following layer)

For output layer units: $\delta_{pj} = k O_{pj}(1 - O_{pj}) (t_{pj} - O_{pj})$

For hidden layer units: $\delta_{pj} = k O_{pj}(1 - O_{pj}) \delta_{pk} W_{jk}$

A unit in the output layer determines its activity by following a 2- step procedure.

Step 1: It computes the total weighted input X_j using the formula $X_j = \sum_i y_i W_{ij}$ Where y_i is the activity level of the j^{th} unit in the previous layer.

Step 2: Calculate the activity y_j using sigmoid function of the total weighted input once the activities of all output units have been determined, the network computes the error E

$$y_j = \frac{1}{1+e^{-x_j}}$$

$$E = \frac{1}{2} (y_i - d_i)^2$$

where y_i is the activity level of the j^{th} unit in the top layer and d_j is the desired output of the j^{th} unit.

3 Developing Best Model

In this study, MLP network has been used for the prediction of short term traffic flow. For development of ANN model, 240 data points have been taken, each of which contained 3 categories of vehicles (2 wheeler, 3\4 wheeler, Heavy vehicles). These three categories were considered as input variables, and one hidden layer was considered. Different Artificial Neural Network(ANN) models have been developed on the training data set. In the present study RMSE, MAE values were used to evaluate the performance of the model and predicted results. The specification of all the models has been presented in Table 2. It was observed that model 5 i.e., neural network with 5 hidden neurons has minimum Root Mean Square Error(RMSE), Mean Absolute Error(MAE) values. Hence it is used to forecast the future values.

Table 2. Different Neural Network models' RMSE, MAE values

Model	Hidden layer	Hidden neurons	Two wheelers	Three/Four wheelers	Heavy vehicles
M1	1	1	RMSE 21.367 MAE 11.106	RMSE 22.174 MAE 11.347	RMSE 3.243 MAE 1.941
M2	1	2	RMSE 21.251 MAE 10.850	RMSE 22.095 MAE 11.220	RMSE 3.224 MAE 1.917
M3	1	3	RMSE 21.145 MAE 10.595	RMSE 22.091 MAE 10.996	RMSE 3.220 MAE 1.913
M4	1	4	RMSE 21.144 MAE 10.591	RMSE 22.081 MAE 10.693	RMSE 3.218 MAE 1.912
M5	1	5	RMSE 21.111 MAE 10.510	RMSE 22.063 MAE 10.643	RMSE 3.20 MAE 1.910

4 Graphical Presentation of Results

The plots of actual vs predicted values of the developed model for the training data set is presented in Fig. 2, Fig. 3 and Fig. 4 for two wheeler, three four wheeler and heavy vehicles respectively.

The plots of actual and predicted values of the developed model for the testing data set is presented in Fig. 5, Fig. 6 and Fig. 7 for two wheeler, three four wheeler and heavy vehicles respectively which shows the adequacy of the model obtained by training data set.

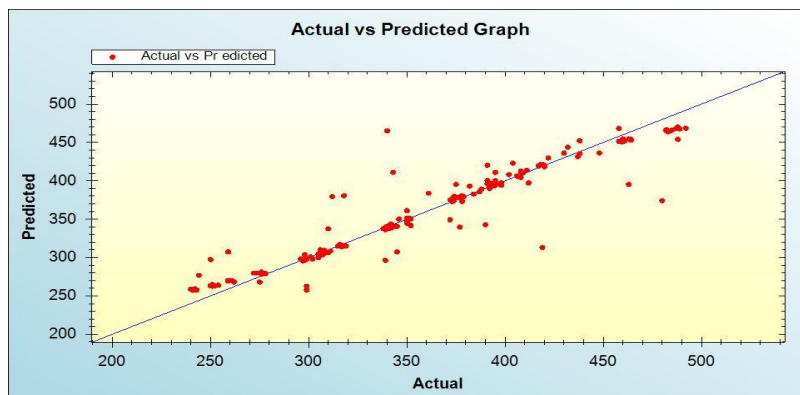


Fig. 2. Plot of actual versus predicted traffic flow for two wheeler vehicles

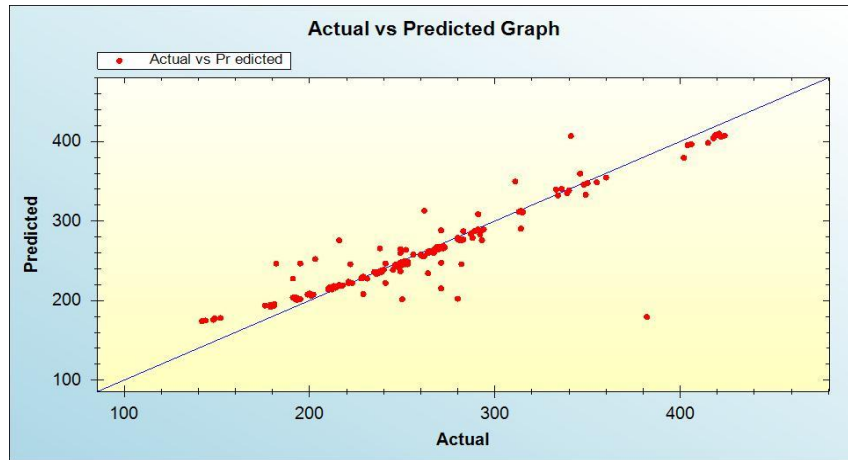


Fig. 3. Plot of actual versus predicted traffic flow for three/four wheeler vehicles

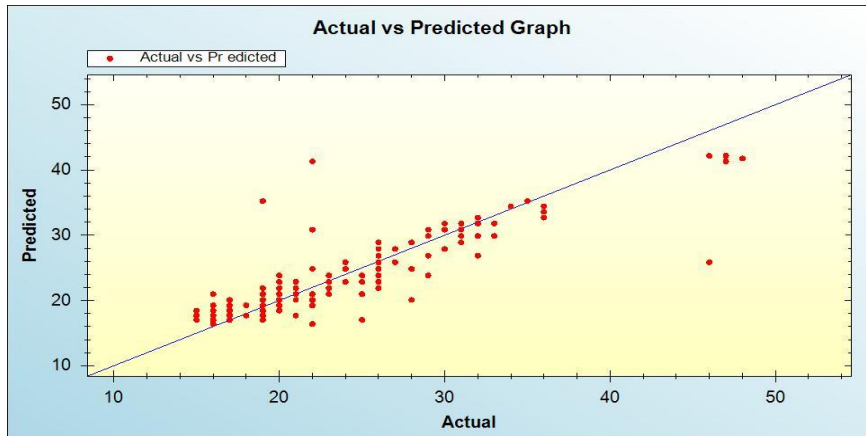


Fig. 4. Plot of actual versus predicted traffic flow for heavy vehicles

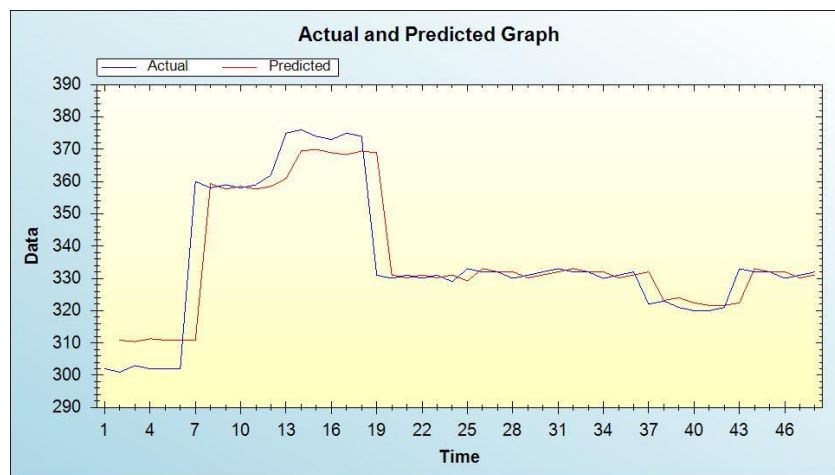


Fig. 5. Plot of actual and predicted traffic flow for two wheeler vehicles (testing data set)

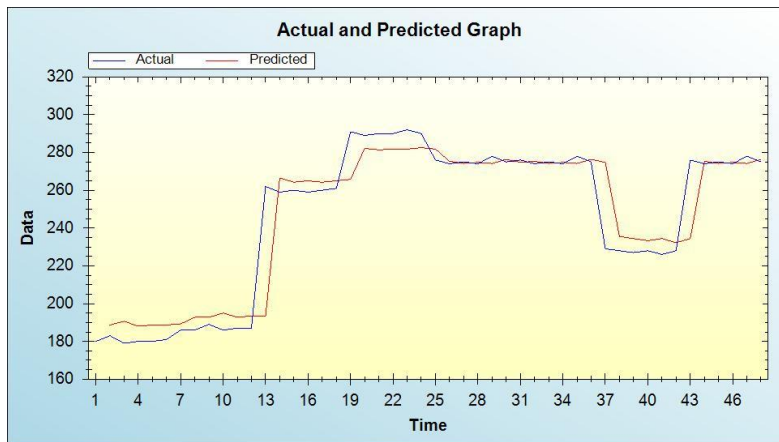


Fig. 6. Plot of actual and predicted traffic flow for three/four wheeler vehicles (testing data set)

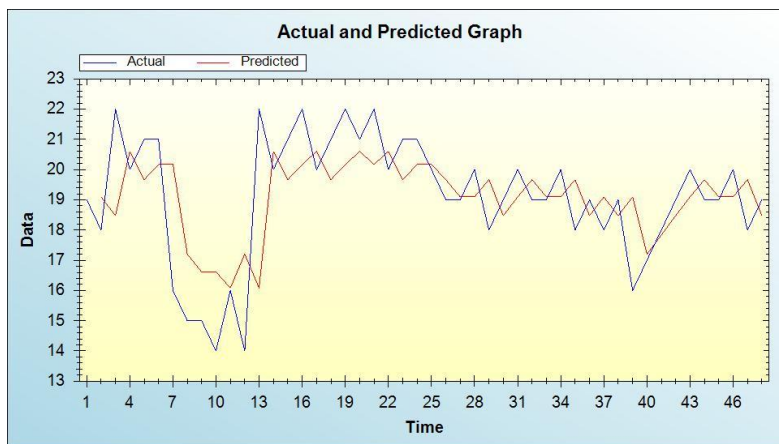


Fig. 7. Plot of actual and predicted traffic flow for heavy vehicles (testing data set)

5 Conclusion

This study presented an Artificial Neural Network(ANN) model to forecast short term traffic volume in a busy junction using MultiLayer Perceptron(MLP) network. The results obtained by the model were quite satisfactory. This study is useful in Intelligent Transport System(ITS) where short term traffic flow forecasting is a major element.

Competing Interests

Authors have declared that no competing interests exist.

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Mean of the Probability Distribution of Departures

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Abstract

This paper proposes the mean for distribution

$f(x) = \begin{cases} 2\mathfrak{S}'P(n, x) - P(n-1, x) & \text{when } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$ for the chosen random variable “how likely there are successive departures in a particular interval”.

Keywords: Random variable; continuous probability distribution; departure rate; density function; mean of the distribution.

1 Introduction

Statistics has the most interesting solutions for the problems in several fields due to its universality. Several new distributions have been developed by taking some subtle transformations on the existing distributions. This paper is continuous work to the previous one [1] which is briefed here. The Random variable of interest is to “how likely there are successive departures in a particular interval” [2-18]. Instead of asking “how many departures take place in a particular time interval (Poisson)”, we ask for “how likely there are successive departures in a particular interval”. Since X is continuous, the PDF should be a function. We had made some inferences about this unknown function. This means the probability distribution that takes into account of measurements those we have surveyed for a considerable period of time. So the output of the inference problem is the distributions of X . We charted the histograms for successive arrivals in a particular interval from which we found the density curves [1,19].

In which case its probability density function is given by

$$f(x) = \begin{cases} 2\mathfrak{S}'P(n, x) - P(n-1, x) & \text{when } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad [1]$$

where $P(n, t)$ is the Truncated Poisson probability distribution of remaining n customers in the queuing system after departing $N-n$ customers from the queuing system in time interval $[0, t]$ [1].

$P(n-1, t)$ is the Truncated Poisson probability distribution of remaining $n-1$ customers in the queuing system after departing $N-(n-1)$ customers from the system in time interval $[0, t]$ and \mathfrak{S}' is the normalizing constant [1].

Graphs of the density function:

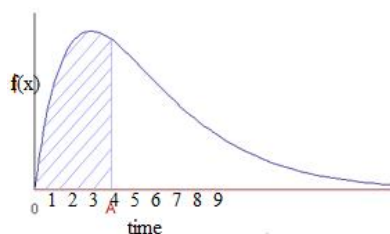


Fig. 1. Density function 1

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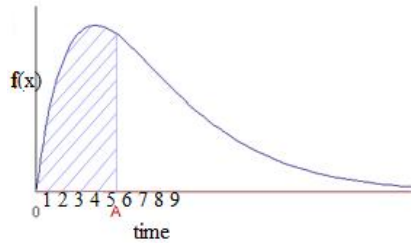


Fig. 2. Density function 2

To extend this distribution theory, the cumulative distributive function and interesting properties of all statistical distributions mean, variance and standard deviation are studied which are widely used in several fields insurance, management, business and finance etc.

2 Mean of the Distribution

$$E(x) = \int_0^{\infty} xf(x)dx \quad [20]$$

When $n = 2$

$$\int_0^{\infty} x\mu [2\tau^1 P(n, x) - P(n-1, x)] dx$$

$$E(x) = \int_0^{\infty} x\mu [2\tau^1 P(2, x) - P(1, x)] dx$$

$$= \int_0^{\infty} \mu x 2\tau^1 \left(\frac{e^{-\mu x} (\mu x)^2}{2} \right) dx - \int_0^{\infty} x\mu e^{-\mu x} \mu x dx$$

$$= \int_0^{\infty} \tau^1 e^{-\mu x} (\mu x)^3 dx - \int_0^{\infty} e^{-\mu x} (\mu x)^2 dx$$

$$\frac{6\tau^1}{\mu} - \frac{2}{\mu}$$

$$E(x) = \frac{6\tau^1 - 2}{\mu}$$

When $n = 3$

$$E(x) = \int_0^{\infty} \frac{2\tau^1 e^{-\mu x} (\mu x)^3}{6} x\mu dx - \int_0^{\infty} \frac{e^{-\mu x} (\mu x)^2}{2} \mu x dx$$

$$= \int_0^{\infty} \frac{\tau^1 e^{-\mu x} \mu^4 x^4}{6} dx - \int_0^{\infty} \frac{e^{-\mu x} \mu^3 x^3}{2} dx$$

$$= \frac{8\tau^1}{\mu} - \frac{3}{\mu}$$

$$E(x) = \frac{8\tau^1 - 3}{\mu}$$

When n = 4

$$\begin{aligned} E(x) &= \int_0^{\infty} x\mu [2\tau^1 P(4, x) - P(3, x)] dx \\ &= \int_0^{\infty} \frac{2\tau^1 e^{-\mu x} (\mu x)^4 \mu x}{24} dx - \int_0^{\infty} \frac{e^{-\mu x} (\mu x)^3 \mu x}{6} dx \\ &= \int_0^{\infty} \frac{\tau^1 e^{-\mu x} x^5 \mu^5}{24} dx - \int_0^{\infty} \frac{e^{-\mu x} \mu^4 x^4}{6} dx \\ &= \frac{10\tau^1}{\mu} - \frac{4}{\mu} \\ E(x) &= \frac{10\tau^1 - 4}{\mu} \end{aligned}$$

	n = 2	n = 3	n = 4	n = N
Mean	$\frac{6\tau^1 - 2}{\mu}$	$\frac{8\tau^1 - 3}{\mu}$	$\frac{10\tau^1 - 4}{\mu}$	$\frac{(2n + 2)\tau^1 - n}{\mu}$

In general we write mean of the above probability distribution is

$$E(x) = \frac{(2n + 2)\tau^1 - n}{\mu}$$

3 Conclusion

We proposed the mean for the Truncated Probability distribution for the assumed random variable. Mean of the above distribution is $E[x] = \frac{(2n + 2)\tau^1 - n}{\mu}$

4 Future Work

There is a lot of scope for future work for this topic. We try to formulate normalizing constant in terms of n. There is scope of formulating Variance, standard deviation and moment generating function of the above distribution.

Competing Interests

Author has declared that no competing interests exist.

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Mean to the Distribution on Arrivals 1

Nirmala Kasturi^{1*}

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Abstract

Several different distributions have been analyzed by a number of authors. There is still more scope for the probability density functions on arrivals which plays major role in lifetime data analysis. This paper proposes the Mean, Variance and Standard Deviation for the density function

$$f(x) = \begin{cases} \lambda [2\mathfrak{Z}(P(n, x) - P(n-1, x))] & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for the chosen random variable "how likely there are successive arrivals" [1,2].}$$

Keywords: Random variable; continuous probability distribution; density function; normalizing constant; mean.

1 Introduction

Due to its universality, Statistics has the significant solutions for the problems in various fields. By taking some inconsiderable subtle transformations on the existing distributions several new distributions have been developed [3-28]. This paper is continuous work to the previous work [1] which briefed here. The Random variable of interest is to "how likely there are successive arrivals in a particular interval". Instead of asking "how many arrivals take place in a particular time interval (Poisson)", we ask for "how likely the system to have successive arrivals in a particular interval of time". Since X is continuous, the PDF should be a function. We had made some inferences about this unknown function. This means the probability distribution that takes into account of measurements those we have surveyed for a considerable period of time. So the output of the inference problem is some distribution of X. We charted the histograms for different number of departures from the system from which we found the density curves [1].

In which case its probability density function is given by

$$f(x) = \begin{cases} \lambda [2\mathfrak{Z}(P(n, x) - P(n-1, x))] & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad [1]$$

Where

P (n, t) is the Poisson probability distribution of arriving n customers in [0, t]

P (n-1, t) is the Poisson probability distribution of arriving n-1 customers in [0, t]

And \mathfrak{Z} is the normalizing constant.

Graph of the density function.

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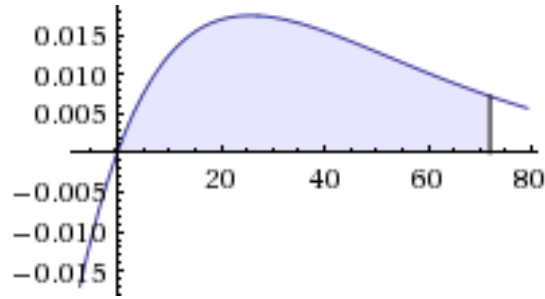


Fig. 1. Density curve - 1

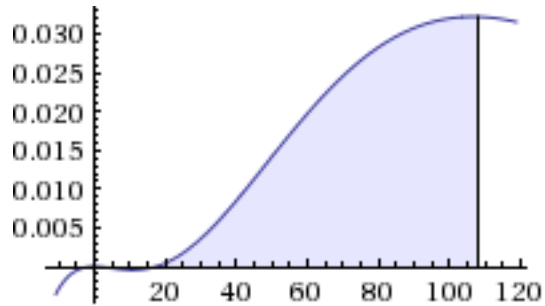


Fig. 2. Density curve - 2

We extend this probability distribution theory by calculating the interesting properties of all statistical distributions, mean, variance and the standard deviation. These properties are studied due to their wide use in several fields like management, insurance, business and finance etc.

2 Mean of the Probability Distribution

Mean of the probability distribution is [29]

$$\begin{aligned} \text{Mean } E(x) &= \int_0^{\infty} xf(x)dx \quad [29] \\ &= \int_0^{\infty} x\lambda [2\tau P(n, x) - P(n-1, x)] dx \end{aligned}$$

When $n = 2$

$$\begin{aligned} \text{Mean } E(x) &= \int_0^{\infty} x\lambda [2\tau P(2, x) - P(1, x)] dx \\ &= \int_0^{\infty} \lambda x 2\tau \left(\frac{e^{-\lambda x} (\lambda x)^2}{2} \right) dx - \int_0^{\infty} x\lambda e^{-\lambda x}, \lambda x dx \\ &= \int_0^{\infty} \tau e^{-\lambda x} (\lambda x)^3 dx - \int_0^{\infty} e^{-\lambda x} (\lambda x)^2 dx \\ &= \frac{6\tau}{\lambda} - \frac{2}{\lambda} \\ E(x) &= \frac{6\tau - 2}{\lambda} \end{aligned}$$

When $n = 3$

$$\begin{aligned} \text{Mean } E(x) &= \int_0^{\infty} \frac{2\tau e^{-\lambda\mu} (\lambda\mu)^3 x\lambda}{6} dx - \int_0^{\infty} \frac{e^{-\lambda\mu} (\lambda x)^2 \lambda x}{2} dx \\ &= \int_0^{\infty} \frac{\tau e^{-\lambda\mu} \lambda^4 x^4}{6} dx - \int_0^{\infty} \frac{e^{-\lambda\mu} \lambda^3 x^3}{2} dx \\ &= \frac{8\tau}{\lambda} - \frac{3}{\lambda} \\ E(x) &= \frac{8\tau - 3}{\lambda} \end{aligned}$$

When $n = 4$

$$\begin{aligned} \text{Mean} = E(x) &= \int_0^{\infty} x\lambda [2\tau P(4, x) - P(3, x)] dx \\ &= \int_0^{\infty} \frac{2\tau e^{-\lambda x} (\lambda x)^4 x\lambda}{24} dx - \int_0^{\infty} \frac{e^{-\lambda x} (\lambda x)^3 x\lambda}{6} dx \\ &= \int_0^{\infty} \frac{\tau e^{-\lambda x} \lambda^5 x^5}{24} dx - \int_0^{\infty} \frac{e^{-\lambda x} \lambda^4 x^4}{6} dx \\ &= \frac{10\tau}{\lambda} - \frac{4}{\lambda} \\ E(x) &= \frac{10\tau - 4}{\lambda} \end{aligned}$$

In general we write mean of the above probability distribution is

$$E(x) = \frac{(2n+2)\tau - n}{\lambda}$$

	n=2	n=3	n=4	n=N
Mean	$\frac{6\tau - 2}{\lambda}$	$\frac{8\tau - 3}{\lambda}$	$\frac{10\tau - 3}{\lambda}$	$\frac{(2n+2)\tau - n}{\lambda}$

3 Conclusion

This paper proposed the Mean for the probability distribution

$$f(x) = \begin{cases} \lambda [2\mathfrak{Z}(P(n, x) - P(n-1, x))] & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for the random variable of the successive arrivals.}$$

Mean of the above distribution is $E[x] = \frac{(2n+2)\tau - n}{\lambda}$

4 Future Work

There is a lot of scope for future work for this topic. We try to formulate normalizing constant in terms of n . There is scope of formulating Variance, standard deviation and moment generating function of the above distribution.

Competing Interests

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